Root Separation Bounds

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Problem

Problem: Univariate

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In : $f \in \mathbb{C}[x]$, square-free

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Out : $B \leq R$ where

- $R = \min_{i \neq j} |\alpha_i - \alpha_j|$ (the root separation)
- α_i are the complex roots of f.

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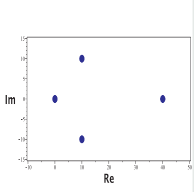
In : $f = x^4 - 60x^3 + 1000x^2 - 8000x$



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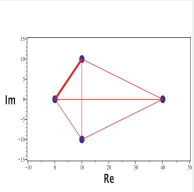

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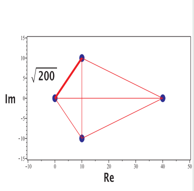

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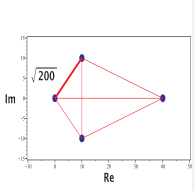

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In : $F = \{f_1, f_2\}$ where

[●] $f_1 = (x_1 - 2)^2 + (x_2 - 5)^2 - 100$

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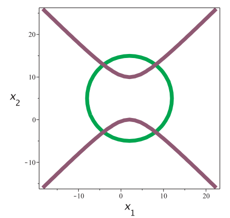

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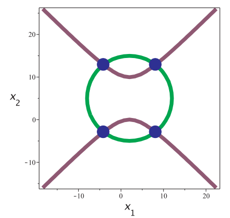

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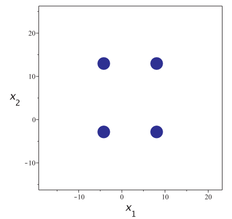

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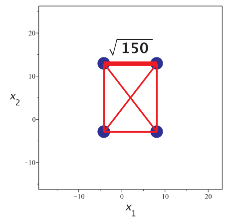

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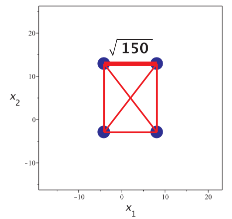

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$$B_{DMM}(F) = \frac{\sqrt{|dis(T_{f_0})|}}{\left(\prod_{i=1}^{n} ||f_i||_\infty^{M_i}\right)^{D-1}} P(d_1, \ldots, d_n, n)$$

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where

$$P(d_1, \ldots, d_n, n) = \cdots, \quad D = \prod_i d_i, \quad M_i = \prod_{j \neq i} d_i$$

$$f_0(u) = \text{a specially chosen separating element.}$$

$$T_{f_0} = \text{the resultant of } (f_0, F) \text{ which eliminates } x_1, \ldots, x_n.$$

Motivation

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Asymptotic analysis of computing time

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Asymptotic analysis of computing time

Sign evaluation of algebraic expressions

Root isolation

Algebraic number theory

Topological analysis of curves

Quantifier elimination

. . .

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Bounds	1964	Mahler
	1973	Mignotte (also, 1988,1995,2000..)
	1974	Collins (also 2001), Horowitz
	1979	Rump
	2000	Mehlorn, Schirra
	2004	Bughead, Mignotte, Sasaki
	2006	Schonhage,Tsigaridas, Emiris
	2008	Muresan, Batra
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Applications	2006	Emeris, Tsigaridas
	2007	Du, Sharma, Yap
	2010	Emiris, Mourrain, Tsigaridas
	2011	Strzebonksi, Tsigaridas
	2012	Tsigaridas, Burr
	2014	Burr

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“Covariant under scaling”	$B(f(x/s))$	$=$	$sB(f(x))$
”Efficiently computable”	Time(B)	$\approx$	Time(B_{Prev})

New Bounds

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Theorem (Herman, Hong, 2016)

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- B_{New} *is covariant under scaling.*
- H *can be computed in in* $\mathcal{O}(n \cdot m + d \log(d))$ *algebraic operations, where*

$$m = \#\ \text{monomials of } F, \qquad d = \sum_{i=1}^{n} d_i$$

Comparisons

Comparison - Magnitude: Univariate

Comparison - Magnitude: Univariate

Bounds

Mahler-Mignotte	**New**

Comparison - Magnitude: Univariate

Bounds

Mahler-Mignotte

New

$$B_{MM} = \frac{\sqrt{3|dis(f)|}}{d^{d/2+1}\ {||f||_2}^{d-1}}$$

Comparison - Magnitude: Univariate

Bounds

Mahler-Mignotte

$$B_{MM} = \frac{\sqrt{3|dis(f)|}}{d^{d/2+1} \; ||f||_2{}^{d-1}}$$

New

$$B_{New} = \frac{\sqrt{3|dis(f)|}}{d^{d/2+1} \; H^{d-1}}$$

Comparison - Magnitude: Univariate

Bounds

Mahler-Mignotte

$$B_{MM} = \frac{\sqrt{3|dis(f)|}}{d^{d/2+1}\ ||f||_2{}^{d-1}}$$

$$||f||_2 = \sqrt{\sum_i |a_i|^2}$$

New

$$B_{New} = \frac{\sqrt{3|dis(f)|}}{d^{d/2+1}\ H^{d-1}}$$

Comparison - Magnitude: Univariate

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Mahler-Mignotte

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New

$$B_{New} = \frac{\sqrt{3|dis(f)|}}{d^{d/2+1}\ H^{d-1}}$$

$$H = \sqrt{\sum_i\ \tilde{s}^{h(i)}\ |a_i|^2}$$

Comparison - Magnitude: Univariate

Bounds

Mahler-Mignotte

$$B_{MM} = \frac{\sqrt{3|dis(f)|}}{d^{d/2+1} \; ||f||_2{}^{d-1}}$$

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$$B_{New} = \frac{\sqrt{3|dis(f)|}}{d^{d/2+1} \; H^{d-1}}$$

$$H = \sqrt{\sum_i \; \tilde{s}^{h(i)} \; |a_i|^2}$$

Example

$f = x^4 - 60x^3 + 1000x^2 - 8000x$

Mahler-Mignotte

New

Comparison - Magnitude: Univariate

Bounds

Mahler-Mignotte

$$B_{MM} = \frac{\sqrt{3|dis(f)|}}{d^{d/2+1} \; ||f||_2{}^{d-1}}$$

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Example

$f = x^4 - 60x^3 + 1000x^2 - 8000x$

Mahler-Mignotte

$$B_{MM} = 8.26 \times 10^{-6}$$

New

Comparison - Magnitude: Univariate

Bounds

Mahler-Mignotte

$$B_{MM} = \frac{\sqrt{3|dis(f)|}}{d^{d/2+1} \; {||f||_2}^{d-1}}$$

$$||f||_2 = \sqrt{\sum_i \; |a_i|^2}$$

New

$$B_{New} = \frac{\sqrt{3|dis(f)|}}{d^{d/2+1} \; H^{d-1}}$$

$$H = \sqrt{\sum_i \; \tilde{s}^{h(i)} \; |a_i|^2}$$

Example

$$f = x^4 - 60x^3 + 1000x^2 - 8000x$$

Mahler-Mignotte

$$B_{MM} = 8.26 \times 10^{-6}$$

New

$$B_{New} = 2.02 \times 10^{-2}$$

Comparison - Magnitude: Univariate

Bounds

Mahler-Mignotte

$$B_{MM} = \frac{\sqrt{3|dis(f)|}}{d^{d/2+1} \; ||f||_2{}^{d-1}}$$

$$||f||_2 = \sqrt{\sum_i \; |a_i|^2}$$

New

$$B_{New} = \frac{\sqrt{3|dis(f)|}}{d^{d/2+1} \; H^{d-1}}$$

$$H = \sqrt{\sum_i \; \tilde{s}^{h(i)} \; |a_i|^2}$$

Example

$f = x^4 - 60x^3 + 1000x^2 - 8000x$

Mahler-Mignotte

$$B_{MM} = 8.26 \times 10^{-6}$$

$$||f||_2 = 8.06 \times 10^3$$

New

$$B_{New} = 2.02 \times 10^{-2}$$

Comparison - Magnitude: Univariate

Bounds

Mahler-Mignotte

$$B_{MM} = \frac{\sqrt{3|dis(f)|}}{d^{d/2+1}\ ||f||_2{}^{d-1}}$$

$$||f||_2 = \sqrt{\sum_i \ |a_i|^2}$$

New

$$B_{New} = \frac{\sqrt{3|dis(f)|}}{d^{d/2+1}\ H^{d-1}}$$

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Example

$f = x^4 - 60x^3 + 1000x^2 - 8000x$

Mahler-Mignotte

$B_{MM} = 8.26 \times 10^{-6}$

$||f||_2 = 8.06 \times 10^3$

New

$B_{New} = 2.02 \times 10^{-2}$

$H = 5.98 \times 10^2$

Comparison - Magnitude: Univariate

Comparison - Magnitude: Univariate

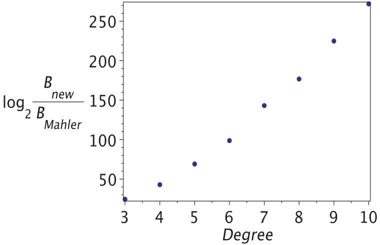
$d = \text{degree}$

$B\text{-height} = max_i \left(|a_i| / \binom{d}{i} \right)^{\frac{1}{d-i}}$

Comparison - Magnitude: Univariate

d = degree

$$B\text{-height} = max_i \left(|a_i| / \binom{d}{i} \right)^{\frac{1}{d-i}}$$

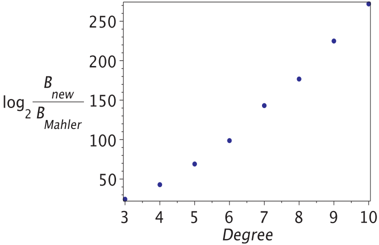


Fix B-height $= 2^7$. Change d.

Comparison - Magnitude: Univariate

d = degree

$$B\text{-height} = max_i \left(|a_i| / \binom{d}{i} \right)^{\frac{1}{d-i}}$$



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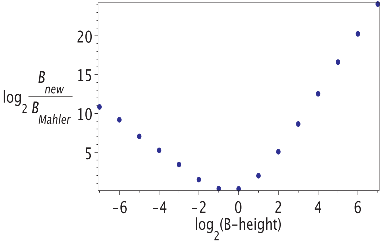

Fix $d = 3$. Change B-height.

Comparison - Behavior Under Scaling: Univariate

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The new bound scales covariantly.

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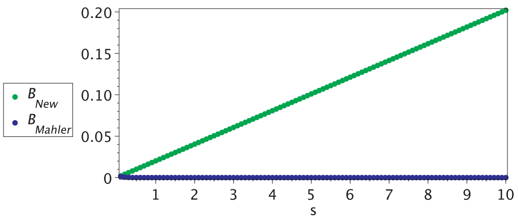
$B_{New}(\, f(x/s)\,)$ [-]

$B_{MM}(\, f(x/s)\,)$ [-]



Comparison - Magnitude: Multivariate

Comparison - Magnitude: Multivariate

Bounds

Davenport-Mahler-Mignotte	**New**

Comparison - Magnitude: Multivariate

Bounds

Davenport-Mahler-Mignotte **New**

$$B_{DMM} = \frac{\sqrt{|dis(T_{f_0})|}}{\left(\prod_{i=1}^{n} ||f_i||_\infty^{M_i}\right)^{D-1}} P$$

Comparison - Magnitude: Multivariate

Bounds

Davenport-Mahler-Mignotte

$$B_{DMM} = \frac{\sqrt{|dis(T_{f_0})|}}{\left(\prod_{i=1}^{n} ||f_i||_\infty^{M_i}\right)^{D-1}} P$$

New

$$B_{New} = \frac{\sqrt{|dis(T_{f_0})|}}{H^{D-1}} P$$

Comparison - Magnitude: Multivariate

Bounds

Davenport-Mahler-Mignotte **New**

$$B_{DMM} = \frac{\sqrt{|dis(T_{f_0})|}}{\left(\prod_{i=1}^{n} ||f_i||_{\infty}^{M_i}\right)^{D-1}} P \qquad B_{New} = \frac{\sqrt{|dis(T_{f_0})|}}{H^{D-1}} P$$

Example

$$\begin{aligned} f_1 &= (x_1-2)^2 + (x_2-5)^2 - 100 \\ f_2 &= (x_2-5)^2 - (x_1-2)^2 - 25 \end{aligned}$$

Davenport-Mahler-Mignotte **New**

Comparison - Magnitude: Multivariate

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Davenport-Mahler-Mignotte **New**

$$B_{DMM} = 2.12 \times 10^{-37}$$

Comparison - Magnitude: Multivariate

Bounds

Davenport-Mahler-Mignotte **New**

$$B_{DMM} = \frac{\sqrt{|dis(T_{f_0})|}}{\left(\prod_{i=1}^{n} ||f_i||_\infty^{M_i}\right)^{D-1}} P \qquad B_{New} = \frac{\sqrt{|dis(T_{f_0})|}}{H^{D-1}} P$$

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Davenport-Mahler-Mignotte **New**

$$B_{DMM} = 2.12 \times 10^{-37} \qquad B_{New} = 2.72 \times 10^{-25}$$

Comparison - Magnitude: Multivariate

Bounds

Davenport-Mahler-Mignotte **New**

$$B_{DMM} = \frac{\sqrt{|dis(T_{f_0})|}}{\left(\prod_{i=1}^{n} ||f_i||_\infty^{M_i}\right)^{D-1}} P \qquad B_{New} = \frac{\sqrt{|dis(T_{f_0})|}}{H^{D-1}} P$$

Example

$$\begin{aligned} f_1 &= (x_1-2)^2 + (x_2-5)^2 - 100 \\ f_2 &= (x_2-5)^2 - (x_1-2)^2 - 25 \end{aligned}$$

Davenport-Mahler-Mignotte **New**

$$B_{DMM} = 2.12 \times 10^{-37} \qquad B_{New} = 2.72 \times 10^{-25}$$

$$|f_1||_\infty^2 ||f_2||_\infty^2 = 5.04 \times 10^5$$

Comparison - Magnitude: Multivariate

Bounds

Davenport-Mahler-Mignotte **New**

$$B_{DMM} = \frac{\sqrt{|dis(T_{f_0})|}}{\left(\prod_{i=1}^{n} ||f_i||_\infty^{M_i}\right)^{D-1}} P \qquad B_{New} = \frac{\sqrt{|dis(T_{f_0})|}}{H^{D-1}} P$$

Example

$$\begin{aligned} f_1 &= (x_1-2)^2 + (x_2-5)^2 - 100 \\ f_2 &= (x_2-5)^2 - (x_1-2)^2 - 25 \end{aligned}$$

Davenport-Mahler-Mignotte **New**

$B_{DMM} = 2.12 \times 10^{-37}$ $B_{New} = 2.72 \times 10^{-25}$

$|f_1||_\infty^2||f_2||_\infty^2 = 5.04 \times 10^5$ $H = 4.64 \times 10^1$

Comparison - Magnitude: Multivariate

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The improvement of the new bound is related to the size of the roots.

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$$||f||_B = \max_{e \in Support(\text{trailing polynomial of } f)} \frac{|a_e|}{\binom{d}{e}}^{\frac{1}{d-|e|}}$$

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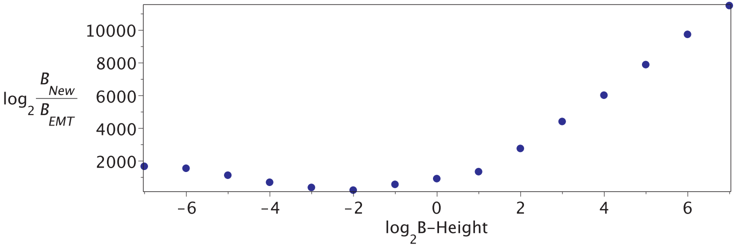

Comparison - Behavior Under Scaling: Multivariate

Comparison - Behavior Under Scaling: Multivariate

The new bound scales covariantly.

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Example

$$\begin{aligned} f_1 &= (x_1-2)^2+(x_2-5)^2-100 \\ f_2 &= (x_2-5)^2-(x_1-2)^2-25 \end{aligned}$$

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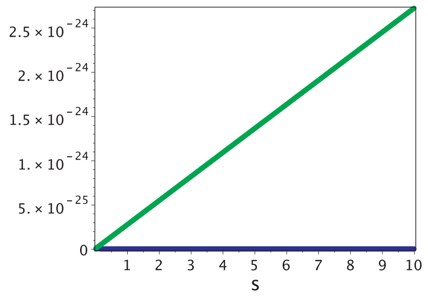
$B_{New}(\, F(x_1/s, x_2/s)\,)$ [-]

$B_{DMM}(\, F(x_1/s, x_2/s)\,)$ [-]



Derivation - Sketch

Derivation: Univariate

Derivation: Univariate

Main Observation

For all $s > 0$

$$\frac{B(f(x/s))}{s}$$

is a root separation bound.

Derivation: Univariate

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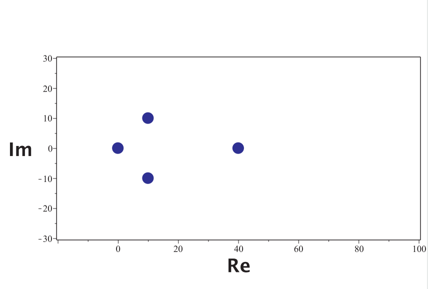
Example

Roots of f [●]



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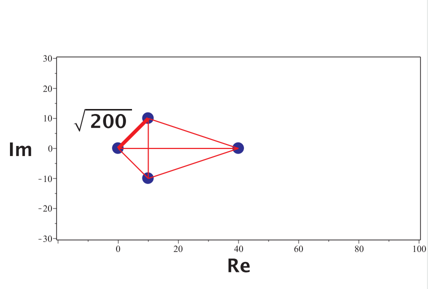

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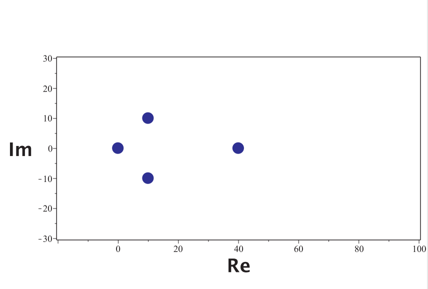
Roots of f [●]



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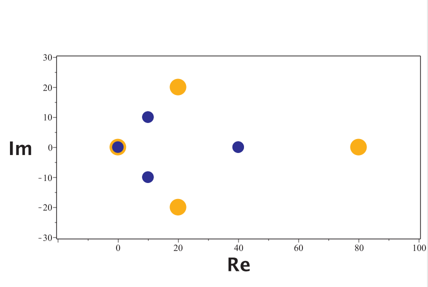
Example

Roots of f [●]

Roots of $f(x/2)$ [●]



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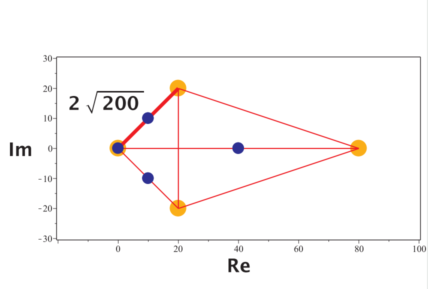
Example

Roots of f [•]

Roots of $f(x/2)$ [•]



Derivation: Univariate

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Main Observation

Let

Derivation: Univariate

Main Observation

Let

$B \quad : f \mapsto \mathbb{R}$ be a root separation bound

Derivation: Univariate

Main Observation

Let

$B \quad : f \mapsto \mathbb{R}$ be a root separation bound

$B^* \quad : f \mapsto \max_s \frac{B(f(x/s))}{s}$

Derivation: Univariate

Main Observation

Let

$$\begin{aligned} B &: f \mapsto \mathbb{R} \text{ be a root separation bound} \\ B^* &: f \mapsto \max_s \frac{B(f(x/s))}{s} \end{aligned}$$

Then

Derivation: Univariate

Main Observation

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Then

B^* is a root separation bound

Derivation: Univariate

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$$B^*(f) \quad \geq \quad B(f)$$

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Example

$f = x^4 - 60x^3 + 1000x^2 - 8000x$

$B = B_M$

Derivation: Univariate

Main Observation

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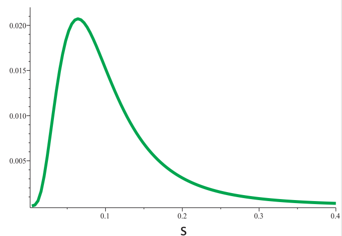
$\frac{B_M(s)}{s}$ [-]



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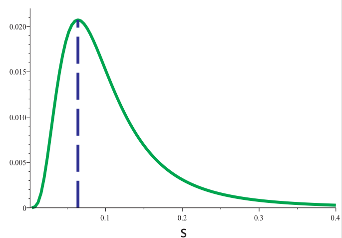
$\frac{B_M(s)}{s}$ [-]

s^* [-]



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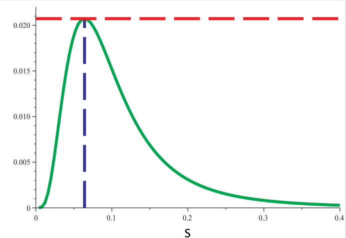
s^* [-]

B^* [-]



Derivation: Univariate

Derivation: Univariate

We can reduce

$$\frac{d}{ds}\frac{B_M(f(x/s))}{s} = 0$$

$$\vdots$$

$$\Updownarrow$$

$$\vdots$$

$$Q(s) = 0$$

Derivation: Univariate

We can reduce

$$\frac{d}{ds}\frac{B_M(f(x/s))}{s} = 0$$

$$\vdots$$

$$\Updownarrow$$

$$\vdots$$

$$Q(s) = 0$$

$Q(s)$ is a polynomial with single sign change in the coefficients

Example

$$Q(s) \;=\; \tfrac{14}{3}\,|a_0|^2\,s^8 + \tfrac{8}{3}\,|a_1|^2\,s^6 + \tfrac{2}{3}\,|a_2|^2\,s^4 - \tfrac{4}{3}\,|a_3|^2\,s^2 - \tfrac{10}{3}\,|a_4|^2$$

Derivation: Univariate

Theorem (Hong 1998)

Let $f = \sum_{i=0}^{m} c_i x^{e_i}$. *Let*

$$\mathcal{H}(f) = \max_{\substack{q \\ c_q<0}} \; \min_{\substack{p \\ c_p>0, e_p>e_q}} \left(\frac{|c_q|}{|c_p|}\right)^{\frac{1}{e_p - e_q}}$$

Then $2\mathcal{H}(f)$ *is an upper bound for the positive real roots of* f.

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Then $2\mathcal{H}(f)$ *is an upper bound for the positive real roots of* f.

Theorem (Herman, Hong, 2014)

Let f *be with a single sign change and let* x^* *be the unique positive root of* f. *Then*

$$1/2\,\mathcal{H}(f) \;\leq\; x^* \;\leq\; 2\,\mathcal{H}(f)$$

Derivation: Univariate

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Theorem (Melhorn, Ray, JSC 2010)

$\mathcal{H}(f)$ *can be computed in linear time in* m.

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$$f = x^4 - 60x^3 + 1000x^2 - 8000x$$

Derivation: Univariate

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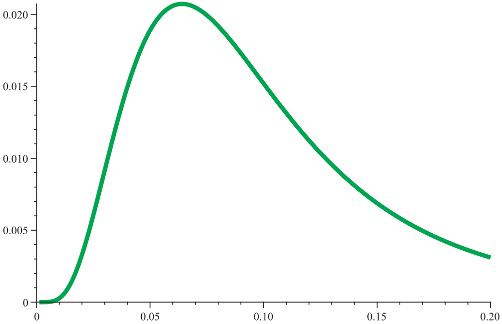
$$\frac{B_M(s)}{s}$$

[-]



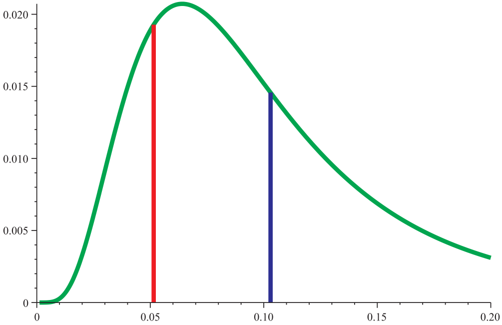
Derivation: Univariate

$$f = x^4 - 60x^3 + 1000x^2 - 8000x$$

$\frac{B_M(s)}{s}$ [-]

$1/2\mathcal{H}(Q)$ [-]

$2\mathcal{H}(Q)$ [-]



Derivation: Univariate

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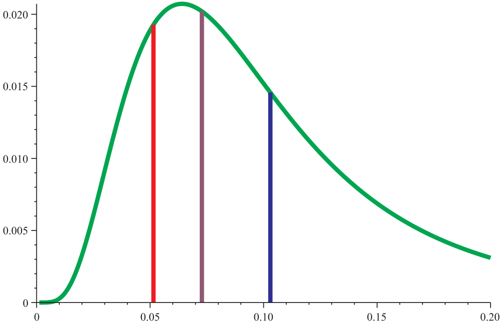
$\frac{B_M(s)}{s}$ [-]

$1/2\mathcal{H}(Q)$ [-]

$2\mathcal{H}(Q)$ [-]

$\tilde{s} = \mathcal{H}(Q)$ [-]



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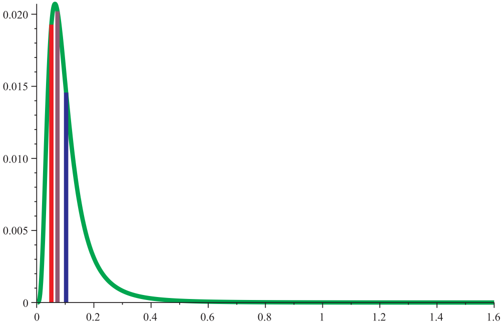
$$f = x^4 - 60x^3 + 1000x^2 - 8000x$$

$\frac{B_M(s)}{s}$ [-]

$1/2\mathcal{H}(Q)$ [-]

$2\mathcal{H}(Q)$ [-]

$\tilde{s} = \mathcal{H}(Q)$ [-]



Derivation: Univariate

- For $B_{New} = \frac{B_M(f(x/\tilde{s}))}{\tilde{s}}$

Derivation: Univariate

- For $B_{New} = \frac{B_M(f(x/\tilde{s}))}{\tilde{s}}$

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  $B_{New}(f) \geq \ B_M(f)$ almost always.

Derivation: Univariate

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  $B_{New}(f) \geq \quad B_M(f)$ almost always.

- We still need to show that B_{new} is covariant under scaling.

Derivation: Univariate

Note

Derivation: Univariate

Note

$$\tilde{s}(f(x/s))$$

Derivation: Univariate

Note

$$\tilde{s}(f(x/s)) \quad = \quad \max_q \min_p \left(\frac{\sqrt{|h(p)|}}{\sqrt{|h(q)|}} \frac{|a_q|/s^q}{|a_p|/s^p} \right)^{\frac{1}{q-p}}$$

Derivation: Univariate

Note

$$\begin{aligned}\tilde{s}(f(x/s)) &= \max_q \min_p \left(\frac{\sqrt{|h(p)|}}{\sqrt{|h(q)|}} \frac{|a_q|/s^q}{|a_p|/s^p} \right)^{\frac{1}{q-p}} \\ &= \max_q \min_p \left(\frac{\sqrt{|h(p)|}}{\sqrt{|h(q)|}} \frac{|a_q|}{|a_p|} \right)^{\frac{1}{q-p}} /s\end{aligned}$$

Derivation: Univariate

Note

$$
\begin{aligned}
\tilde{s}(f(x/s)) &= \max_q \min_p \left(\frac{\sqrt{|h(p)|}}{\sqrt{|h(q)|}} \frac{|a_q|/s^q}{|a_p|/s^p} \right)^{\frac{1}{q-p}} \\
&= \max_q \min_p \left(\frac{\sqrt{|h(p)|}}{\sqrt{|h(q)|}} \frac{|a_q|}{|a_p|} \right)^{\frac{1}{q-p}} /s \\
&= \tilde{s}/s
\end{aligned}
$$

Derivation: Univariate

Note

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Hence

Derivation: Univariate

Note

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Hence

$$B_{New}(f(x/s))$$

Derivation: Univariate

Note

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Hence

$$B_{New}(f(x/s)) = \frac{B_M(f(\frac{x}{\tilde{s}(f(x/s))}/s))}{\tilde{s}(f(x/s))}$$

Derivation: Univariate

Note

$$\begin{aligned}
\tilde{s}(f(x/s)) &= \max_q \min_p \left(\frac{\sqrt{|h(p)|}}{\sqrt{|h(q)|}} \frac{|a_q|/s^q}{|a_p|/s^p} \right)^{\frac{1}{q-p}} \\
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Hence

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B_{New}(f(x/s)) &= \frac{B_M(f(\frac{x}{\tilde{s}(f(x/s))}/s))}{\tilde{s}(f(x/s))} \\
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Derivation: Univariate

Note

$$\begin{aligned}\tilde{s}(f(x/s)) &= \max_q \min_p \left(\frac{\sqrt{|h(p)|}}{\sqrt{|h(q)|}} \frac{|a_q|/s^q}{|a_p|/s^p} \right)^{\frac{1}{q-p}} \\ &= \max_q \min_p \left(\frac{\sqrt{|h(p)|}}{\sqrt{|h(q)|}} \frac{|a_q|}{|a_p|} \right)^{\frac{1}{q-p}} /s \\ &= \tilde{s}/s\end{aligned}$$

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Derivation: Univariate

Note

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Hence

$$\begin{aligned}B_{New}(f(x/s)) &= \frac{B_M(f(\frac{x}{\tilde{s}(f(x/s))}/s))}{\tilde{s}(f(x/s))} \\ &= \frac{B_M(f(\frac{x}{\tilde{s}/s}/s))}{\tilde{s}/s} \\ &= s\,\frac{B_M(f(x/\tilde{s})}{\tilde{s}} \\ &= s\,B_{New}(f)\end{aligned}$$

Derivation: Univariate

Note

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We have proved the covariance.

Derivation: Multivariate

Derivation: Multivariate

Main Observation

Let $s > 0$ and

$$B' : F \mapsto \frac{B(\,F^{(s)}\,)}{s}$$

where

$$F^{(s)} = (f_1^{(s)}, \ldots, f_n^{(s)}), \qquad f^{(s)} = s^{\deg(f)} f(x/s)$$

Then B' is a root separation bound.

Derivation: Multivariate

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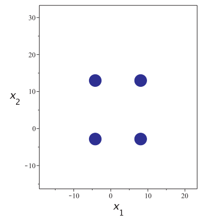
Example

$$f_1 = (x_1 - 2)^2 + (x_2 - 5)^2 - 100$$
$$f_2 = (x_2 - 5)^2 - (x_1 - 2)^2 - 25$$

Roots of F [•]



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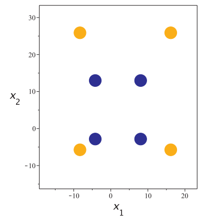
Roots of F [●]

Roots of $F^{(2)}$ [●]



Derivation: Multivariate

Main Observation

Let $s > 0$ and

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where

$$F^{(s)} = (f_1^{(s)}, \ldots, f_n^{(s)}), \qquad f^{(s)} = s^{\deg(f)} f(x/s)$$

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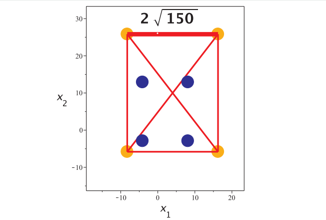
Roots of F [•]

Roots of $F^{(2)}$ [•]



Derivation: Multivariate

Derivation: Multivariate

Proposition

Let

$$B^* : F \mapsto \max_{s>0} \frac{B(F^{(s)})}{s}$$

Then

- B^* is a root separation bound.
- B^* scales covariantly
- $B^*(F) \geq B(F)$

Derivation: Multivariate

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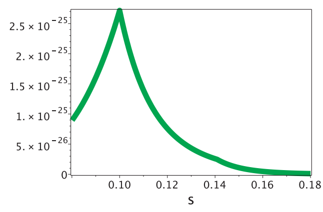
$\frac{B_{DMM}(F^{(s)})}{s}$ [•]



Derivation: Multivariate

Derivation: Multivariate

We have

$$\arg\max_{s>0} \frac{B_{DMM}(F^{(s)})}{s} \; = \; \ldots \; = \arg\min_{s>0} R(s)$$

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Derivation: Multivariate

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where

$$R(s) = \frac{\prod_{i=1}^{n} ||f_i^{(s)}||_\infty^{M_i}}{s^{\frac{D}{2} - \frac{1}{D-1}}}$$

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Example

$$\begin{aligned} f_1 &= (x_1-2)^2+(x_2-5)^2-100 \\ &= x_1^2+x_2^2-4x_1-10x_2-71 \end{aligned}$$

Derivation: Multivariate

We have

$$\arg\max_{s>0} \frac{B_{DMM}(F^{(s)})}{s} \; = \; \dots \; = \; \arg\min_{s>0} R(s)$$

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$$\begin{aligned} f_1 &= (x_1 - 2)^2 + (x_2 - 5)^2 - 100 \\ &= x_1^2 + x_2^2 - 4x_1 - 10x_2 - 71 \\ ||f_1^{(s)}||_\infty &= ||x_1^2 + x_2^2 - 4sx_1 - 10sx_2 - 71s^2||_\infty \end{aligned}$$

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$$\begin{aligned} f_1 &= (x_1 - 2)^2 + (x_2 - 5)^2 - 100 \\ &= x_1^2 + x_2^2 - 4x_1 - 10x_2 - 71 \\ ||f_1^{(s)}||_\infty &= ||x_1^2 + x_2^2 - 4sx_1 - 10sx_2 - 71s^2||_\infty \\ &= \max\{1, 1, 4s, 10s, 71s^2\} \end{aligned}$$

Derivation: Multivariate

We have

$$\arg\max_{s>0} \frac{B_{DMM}(F^{(s)})}{s} \;=\; \dots \;=\; \arg\min_{s>0} R(s)$$

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f_1 &= (x_1-2)^2 + (x_2-5)^2 - 100 \\
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&= \max\{1, 1, 4s, 10s, 71s^2\} \\
f_2 &= (x_2-5)^2 - (x_1-2)^2 - 25 \\
&= x_2^2 - x_1^2 + 4x_1 - 10x_2 - 4
\end{aligned}$$

Derivation: Multivariate

We have

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&= \max\{1, 1, 4s, 10s, 71s^2\} \\
f_2 &= (x_2-5)^2 - (x_1-2)^2 - 25 \\
&= x_2^2 - x_1^2 + 4x_1 - 10x_2 - 4 \\
||f_2^{(s)}||_\infty &= \max\{1, 1, 4s, 10s, 4s^2\}
\end{aligned}$$

Derivation: Multivariate

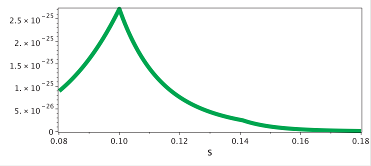
We have

$$\arg\max_{s>0} \frac{B_{DMM}(F^{(s)})}{s} = \dots = \arg\min_{s>0} R(s)$$

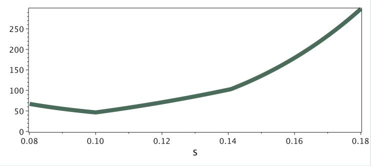
where

$$R(s) = \frac{\prod_{i=1}^{n} ||f_i^{(s)}||_\infty^{M_i}}{s^{\frac{D}{2}-\frac{1}{D-1}}}$$

Example



$$\frac{B_{DMM}(F^{(s)})}{s}$$



$$R(s) = \frac{||f_1^{(s)}||^2 \cdot ||f_2^{(s)}||^2}{s^{\frac{4}{2}-\frac{1}{4-1}}}$$

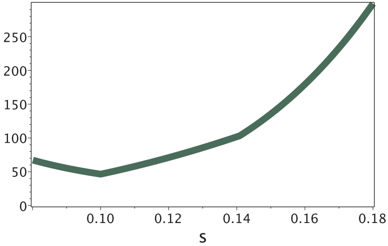
Derivation: Multivariate

Derivation: Multivariate

In $Log - Log$ space, R is piecewise linear.

Derivation: Multivariate

In *Log* − *Log* space, R is piecewise linear.



Derivation: Multivariate

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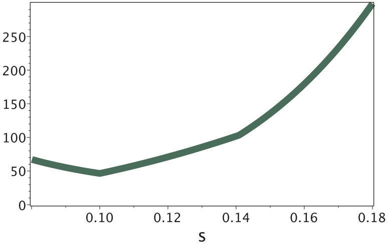
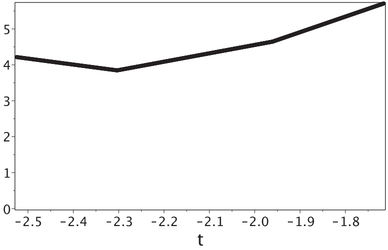


$$\xrightarrow{Log-Log}$$



Derivation: Multivariate

In *Log* – *Log* space, R is piecewise linear.

$$\log(R(s)) = \log\left(\frac{\prod_{i=1}^{n} ||f_i^{(s)}||_\infty^{M_i}}{s^{\frac{D}{2}-\frac{1}{D-1}}}\right)$$

Derivation: Multivariate

In $Log - Log$ space, R is piecewise linear.

$$\begin{aligned}\log(R(s)) &= \log\left(\frac{\prod_{i=1}^{n} ||f_i^{(s)}||_\infty^{M_i}}{s^{\frac{D}{2}-\frac{1}{D-1}}}\right)\\ &= \sum_{i=1}^{n} M_i \log(||f_i^{(s)}||_\infty) - \left(\frac{D}{2} - \frac{1}{D-1}\right)\log(s)\end{aligned}$$

Derivation: Multivariate

In $Log - Log$ space, R is piecewise linear.

$$\begin{aligned}
\log(R(s)) &= \log\left(\frac{\prod_{i=1}^{n} ||f_i^{(s)}||_\infty^{M_i}}{s^{\frac{D}{2}-\frac{1}{D-1}}}\right)\\
&= \sum_{i=1}^{n} M_i \log(||f_i^{(s)}||_\infty) - \left(\frac{D}{2} - \frac{1}{D-1}\right)\log(s)\\
&= \sum_{i=1}^{n} M_i \max_{e\in Support(f)} (d_i - |e|)\log(s) + \log(|a_e|)\\
&\quad - \left(\frac{D}{2} - \frac{1}{D-1}\right)\log(s)
\end{aligned}$$

Derivation: Multivariate

In $Log - Log$ space, R is piecewise linear.

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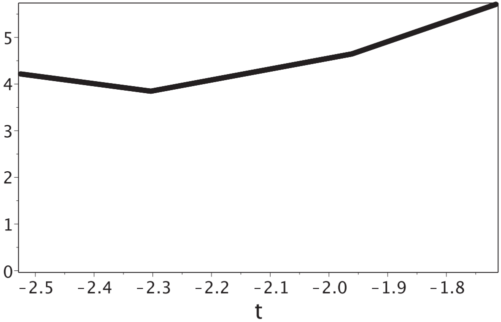
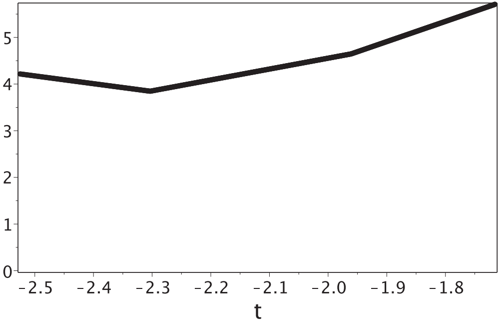
\log(R(s)) &= \log\left(\frac{\prod_{i=1}^{n}||f_i^{(s)}||_\infty^{M_i}}{s^{\frac{D}{2}-\frac{1}{D-1}}}\right)\\
&= \sum_{i=1}^{n} M_i \log(||f_i^{(s)}||_\infty) - \left(\frac{D}{2}-\frac{1}{D-1}\right)\log(s)\\
&= \sum_{i=1}^{n} M_i \max_{e\in Support(f)} (d_i - |e|)\log(s) + \log(|a_e|)\\
&\quad - \left(\frac{D}{2}-\frac{1}{D-1}\right)\log(s)\\
&= \sum_{i=1}^{n} M_i \max_{e\in Support(f)} (d_i - |e|)t + \log(|a_e|)\\
&\quad - \left(\frac{D}{2}-\frac{1}{D-1}\right)t
\end{aligned}$$

Derivation: Multivariate

Derivation: Multivariate

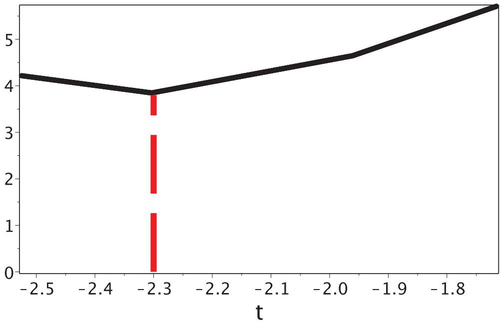


Derivation: Multivariate



$$\log(R)(t) = \begin{cases} -5/3 \cdot t & t \leq -2.30 \\ 7/3 \cdot t + 9.21 & -2.30 \leq t \leq -1.96 \\ 13/3 \cdot t + 13.1 & -1.96 \leq t \leq 0.91 \\ 19/3 \cdot t + 11.3 & 0.91 \leq t \end{cases}$$

Derivation: Multivariate



$$\log(R)(t) = \begin{cases} -5/3 \cdot t & t \leq -2.30 \\ 7/3 \cdot t + 9.21 & -2.30 \leq t \leq -1.96 \\ 13/3 \cdot t + 13.1 & -1.96 \leq t \leq 0.91 \\ 19/3 \cdot t + 11.3 & 0.91 \leq t \end{cases}$$

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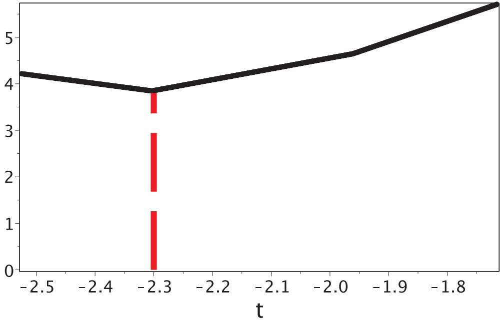
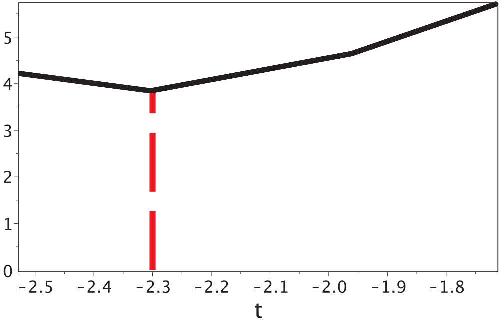


$$\log(R)(t) = \begin{cases} -5/3 \cdot t & t \leq -2.30 \\ 7/3 \cdot t + 9.21 & -2.30 \leq t \leq -1.96 \\ 13/3 \cdot t + 13.1 & -1.96 \leq t \leq 0.91 \\ 19/3 \cdot t + 11.3 & 0.91 \leq t \end{cases}$$

Derivation: Multivariate



$$\log(R)(t) = \begin{cases} -5/3 \cdot t & t \leq -2.30 \\ 7/3 \cdot t + 9.21 & -2.30 \leq t \leq -1.96 \\ 13/3 \cdot t + 13.1 & -1.96 \leq t \leq 0.91 \\ 19/3 \cdot t + 11.3 & 0.91 \leq t \end{cases}$$

Given a “nice” piece-wise description of $\log(R)$, finding the minimum is easy.

Derivation: Multivariate

Derivation: Multivariate

Recall

$$\log(R) = \sum_{i=1}^{n} M_i \log(||f_i^{(s)}||_\infty) - \left(\frac{D}{2} - \frac{1}{D-1}\right)\log(s)$$

Derivation: Multivariate

Recall

$$\log(R) = \sum_{i=1}^{n} M_i \log(||f_i^{(s)}||_\infty) - \left(\frac{D}{2} - \frac{1}{D-1}\right) \log(s)$$

We will find a piece-wise description of $\log(||f_i^{(s)}||_\infty)$ for $i = 1, \ldots, n$, then combine them to find a piece-wise description for $\log(R)$.

Derivation: Multivariate

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We have

$$\log(||f_1^{(s)}||_\infty) = \log(||x_1^2 + x_2^2 - 4sx_1 - 10sx_2 - 71s^2||_\infty)$$

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$$\begin{aligned}\log(||f_1^{(s)}||_\infty) &= \log(||x_1^2 + x_2^2 - 4sx_1 - 10sx_2 - 71s^2||_\infty) \\ &= \log(\max\{1, 1, 4s, 10s, 71s^2\})\end{aligned}$$

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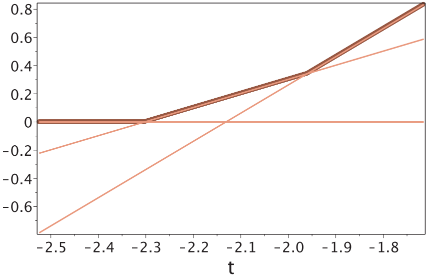

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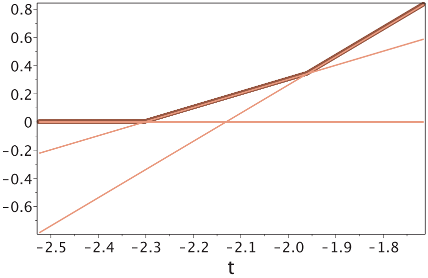

Note

Since $\log(\|f_1^{(s)}\|_\infty)$ has $\mathcal{O}(d_i)$ pieces, we can compute the piece-wise rep. using $\mathcal{O}(d_i)$ comparisons.

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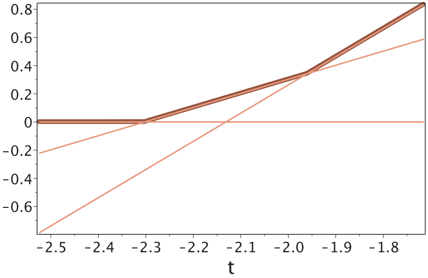

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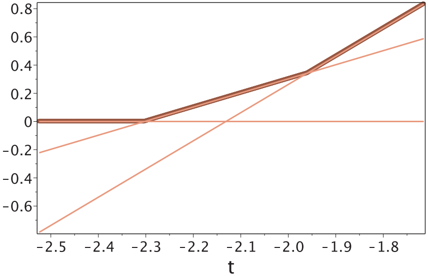

$$\begin{cases} 0 & t \leq -2.30 \\ t + 2.30 & -2.30 \leq t \leq -1.96 \\ 2t + 4.26 & -1.96 \leq t \end{cases}$$

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Theorem

The minimizer of $R(s)$ can be computed in $\mathcal{O}(n \cdot m + d\log(d))$ algebraic operations.

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- Translation invariance
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- Lower bound for discriminant
  - easy for integer coefficients, but they are pessimistic

Thanks!