Geometric constructions and invariance

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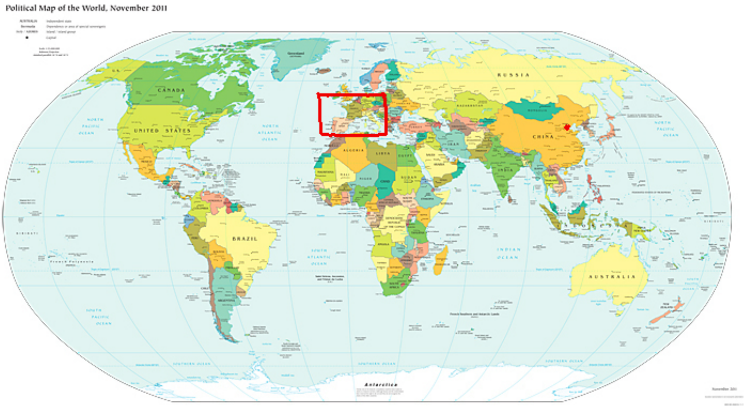
Introduction



Introduction

Geometric constraint solving in geometric design

A practical question: how to design an object?

A simple example:

Geometric constraint solving in geometric design

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A simple example:

The Strasbourg's Cathedral (left)

...

Geometric constraint solving in geometric design

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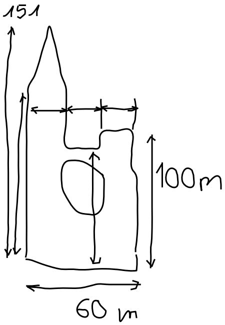
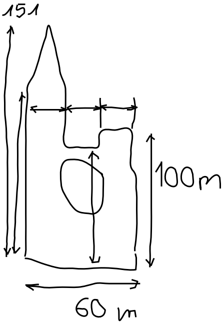
and a first draft ... (below)



Geometric constraint solving in geometric design

A practical question: how to design an object?



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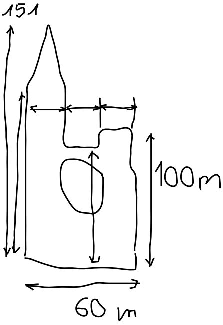

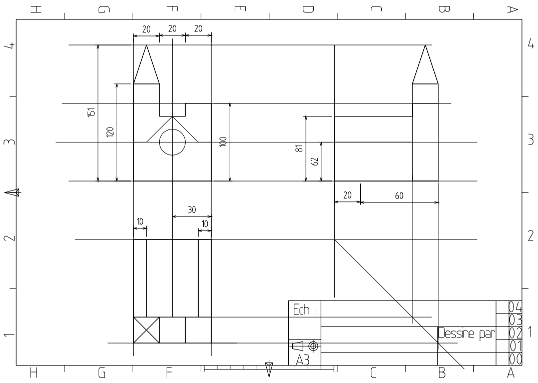

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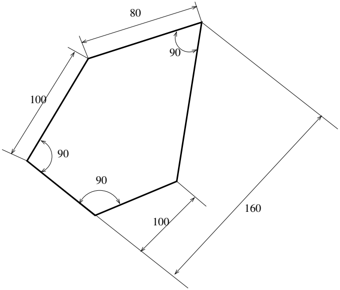

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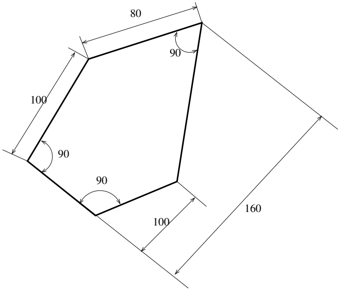

Sketch+measurement →
graphical statement

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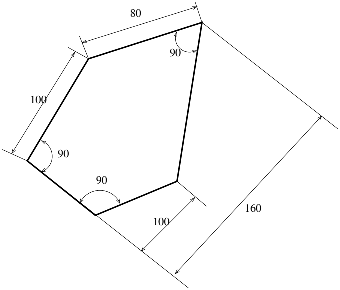

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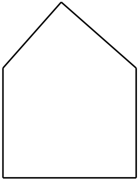

a numerical solution

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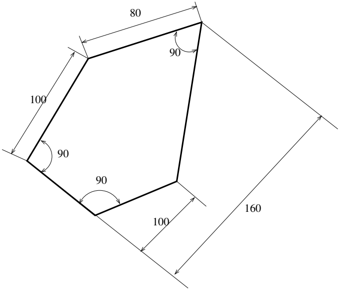

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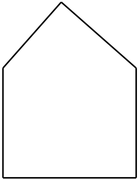

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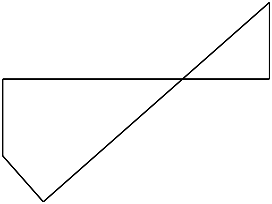

... but not this one

Geometric constraint solving in CAD

Exploit continuity and numerical methods ...

Continuity

- framework: $\mathbb{R}^2$ ou $\mathbb{R}^3$;
- continuous constraints on continuous domains
- for instance the distance bewteen points A B, can by expressed by $(x_A - x_B)^2 + (y_A - y_B)^2 = k^2$

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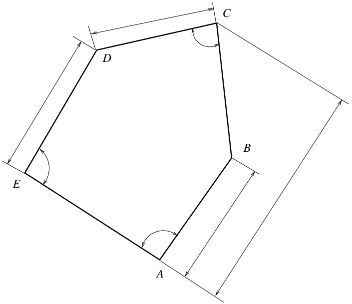

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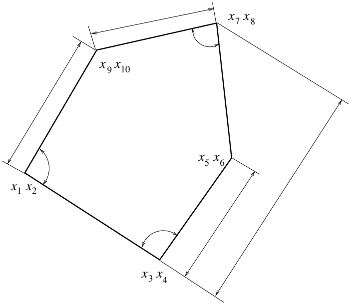

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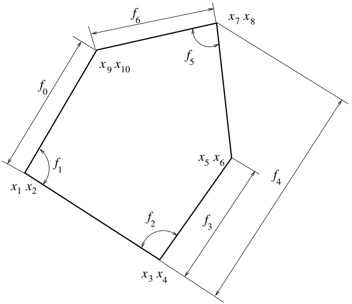

coordinates x_1, x_2, x_9 and x_{10} are fixed

$$\begin{cases} f_1(x_3, x_4) & = 0 \\ f_2(x_3, x_4, x_5, x_6) & = 0 \\ f_3(x_3, x_4, x_5, x_6) & = 0 \\ f_4(x_3, x_4, x_7, x_8) & = 0 \\ f_5(x_5, x_6, x_7, x_8) & = 0 \\ f_6(x_7, x_8) & = 0 \end{cases}$$

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Bingo!

It seems now easy to solve by using for instance

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But ...

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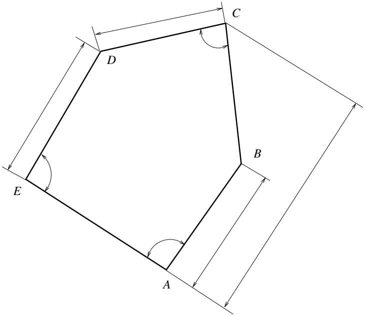
... or use the good old locus method

Indeed, we can draw precisely the figure using geometric reasoning



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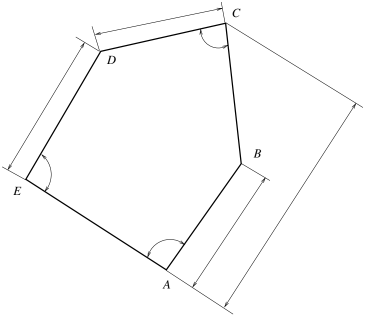

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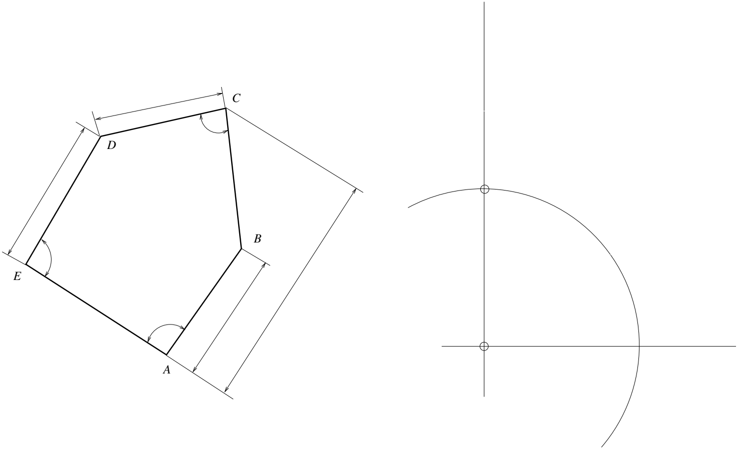

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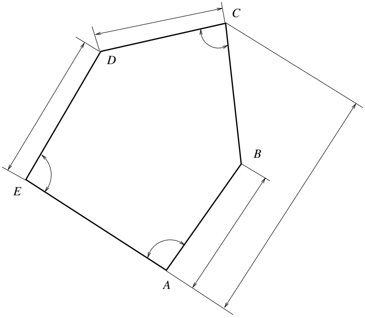

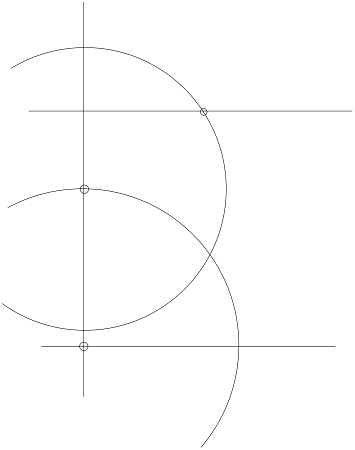

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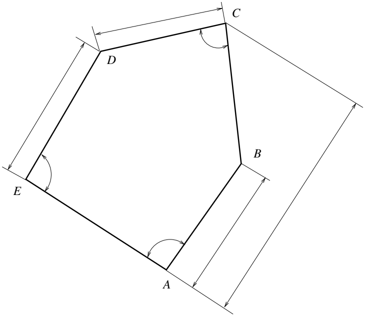

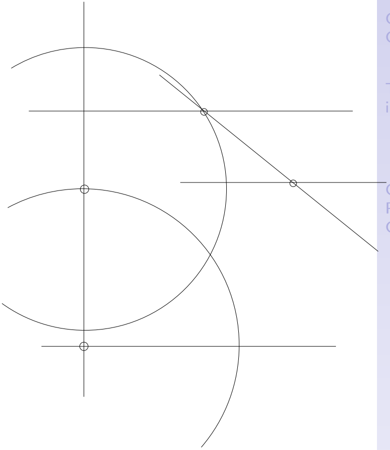

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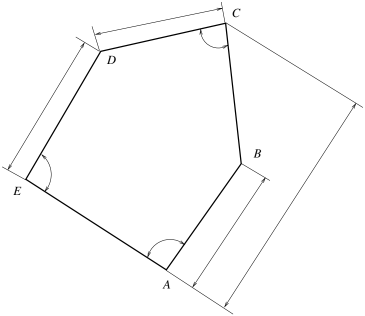

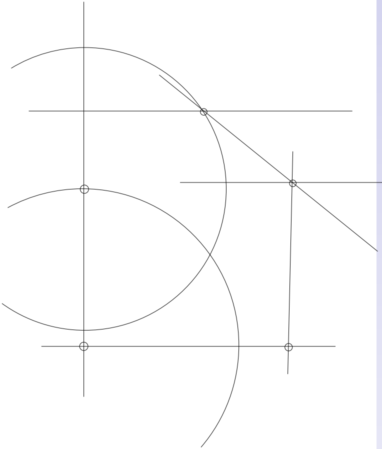

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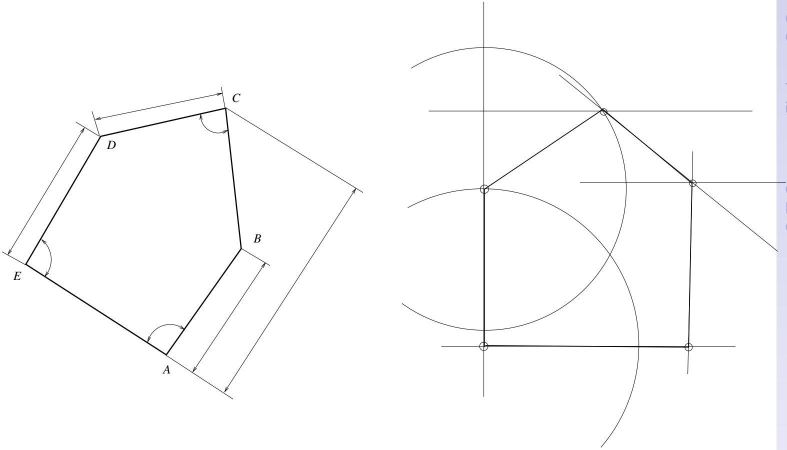
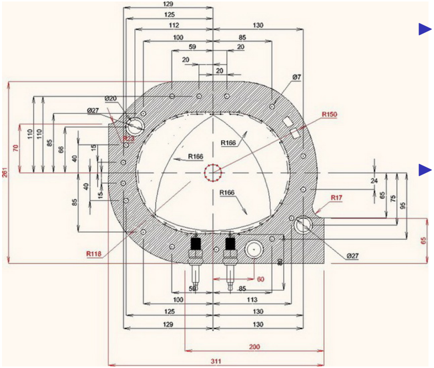


Discussion

Need for a qualitative study of a constraint system



$\sim$ 50 constraints

- specifying is difficult
  - constraintness
  - over-constrained system
- decomposition
  - natural decomposition of a mechanism
  - generalization?

Qualitative study of a system of equations

Qualitative study through Dulmage-Mendelsohn decomposition

This method acts on equational systems without considering the semantics:

Roughly speaking

- triangularization by blocks
- using only permutations on variables
- taking only the structure into account :
- not too much equations, but enough
- the method relies on flows computation in graph theory

Qualitative study through Dulmage-Mendelsohn decomposition

Example of equational system

$$\begin{cases} a_1 x_1^2 x_2 + b_1 x_2 + e_1 x_5 x_1 = 1.25 & (1) \\ d_2 x_4 + e_2 x_5 + f_2 x_6 = 2.22 & (2) \\ b_3 x_2 x_7 + g_3 x_7 = 3.14 & (3) \\ a_4 x_1 + c_4 x_3 + f_4 x_6 = 42 & (4) \\ c_5 x_3 + d_5 x_4 = 5.31 & (5) \\ b_6 x_2^2 + g_6 x_7^2 = 5.89 & (6) \\ a_7 x_2 + e_7 x_2^3 + g_7 x_7 = 7.89 & (7) \end{cases}$$

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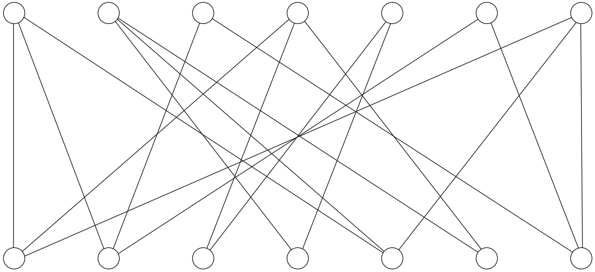
Structural constraintness

- over-constrained : more equations than unknowns
- well-constrained :
  - as many equations as unknowns
  - no over-constrained sub-systems

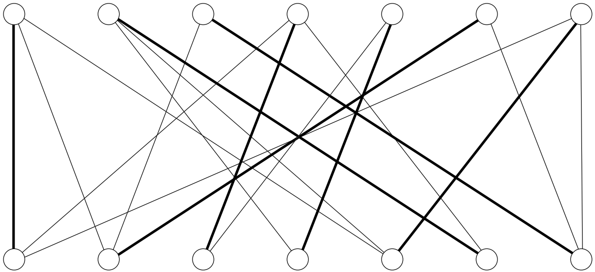
Qualitative study through Dulmage-Mendelsohn decomposition

Bipartite graph equations-unknowns



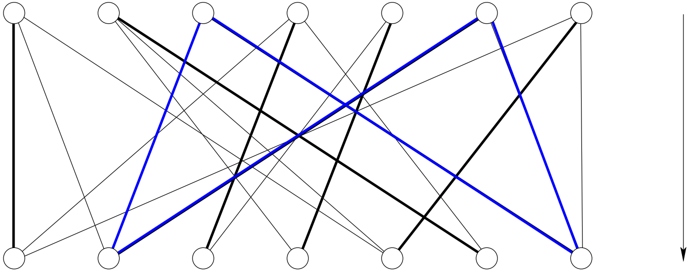
Qualitative study through Dulmage-Mendelsohn decomposition

maximum matching and constraintness (König-Hall)



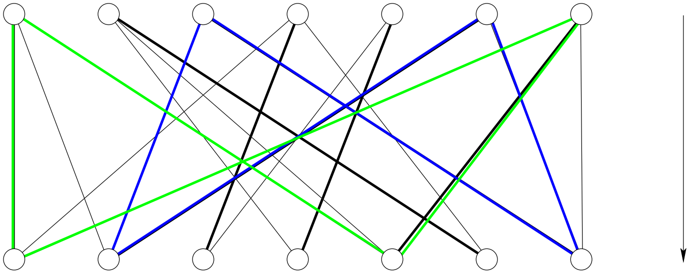
Qualitative study through Dulmage-Mendelsohn decomposition

Orientation - strongly connected components



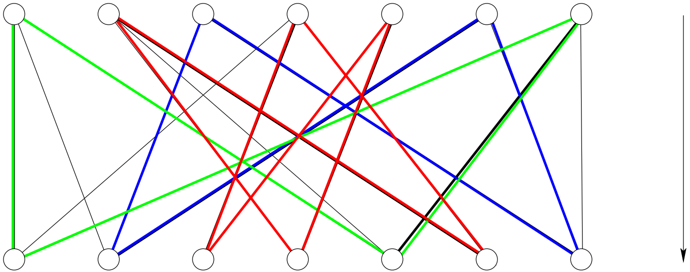
Qualitative study through Dulmage-Mendelsohn decomposition

strongly connected components (2)



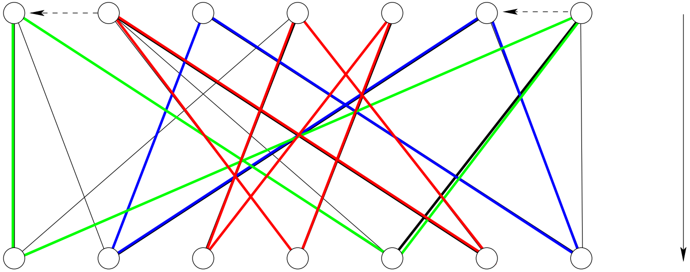
Qualitative study through Dulmage-Mendelsohn decomposition

strongly connected components (3)



Qualitative study through Dulmage-Mendelsohn decomposition

Dependencies



Qualitative study through Dulmage-Mendelsohn decomposition

A re-ordering

$$\begin{cases} b_3x_2x_7 + g_3x_7 & = & 3.14 & (x_2, x_7) \\ b_6x_2^2 + g_6x_7^2 & = & 5.89 & (x_2, x_7) \\ a_1x_1^2x_2 + b_1x_2 + e_1x_5x_1 & = & 1.25 & (x_1, x_5) \\ a_7x_1 + e_7x_5 + g_7x_7 & = & 7.89 & (x_1, x_5) \\ d_2x_4 + e_2x_5 + f_2x_6 & = & 2.22 & (x_3, x_4, x_6) \\ a_4x_1 + c_4x_3 + f_4x_6 & = & 42 & (x_3, x_4, x_6) \\ c_5x_3 + d_5x_4 & = & 5.31 & (x_3, x_4, x_6) \end{cases}$$

Qualitative study through Dulmage-Mendelsohn decomposition

What happens if there are too many ...

▷ equations: there is no perfect matching and a component containing only equations

▷ unknowns: there is no perfect matching and a component containing only unknowns

Dulmage-Mendelson decomposition(s)

- coarse decomposition: 3 components well-, under- and over-constrained.
- fine decomposition: it computes the dependencies in the well-constrained component.

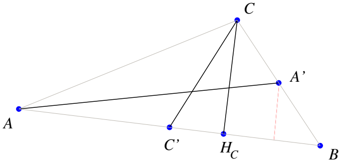
Geometric constructions of triangles

Geometric constructions (example)

Statement

Construct a triangle ABC knowing:

- *points A and A' mid. of $[BC]$*
- *the altitude $[CH_C]$*
- *the median $[CC']$*



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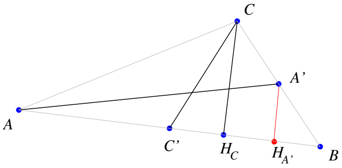

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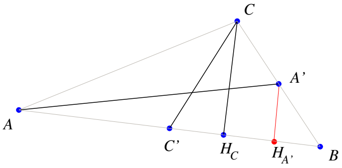

Construction plan

- A and A' are given

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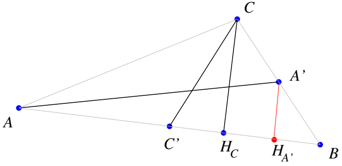

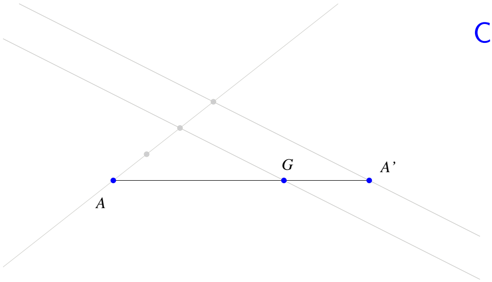

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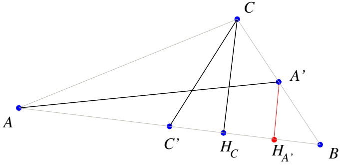

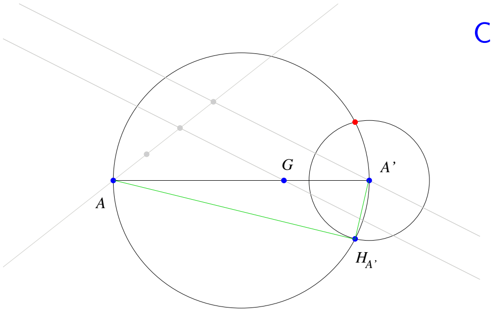

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- construct $H_{A'}$ (2)

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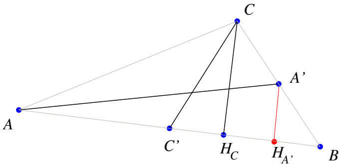

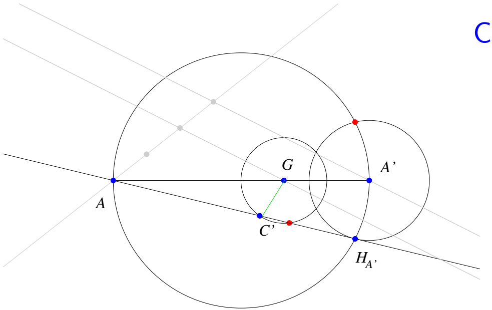

Construction plan

- A and A' are given
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- construct $H_{A'}$ (2)
- construct line (AB)
- construct C' (2)

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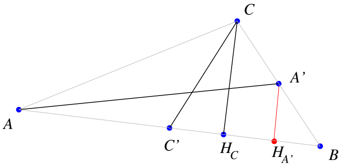
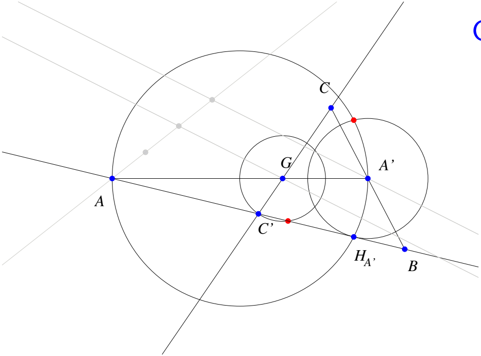
- *the median $[CC']$*





Construction plan

- A and A' are given
- construct G
- construct $H_{A'}$ (2)
- construct line (AB)
- construct C' (2)
- construct B
- construct C

Discussion (again)

- a construction plan can be seen from different points of view
- historically straightedge (ruler) and compass are considered as basic tools
- but other tools are sometimes used, origamis for instance

Discussion (again)

- a construction plan can be seen from different points of view
  - it is an ordering of elementary construction steps
  - it is a kind of decomposition of the problem
  - it expresses an exact solution if all the steps are exact
- historically straightedge (ruler) and compass are considered as basic tools
- but other tools are sometimes used, origamis for instance

Discussion (again)

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Discussion (again)

- a construction plan can be seen from different points of view
- historically straightedge (ruler) and compass are considered as basic tools
  ... leading to famous RC-unconstructible problems
  - trisection of an angle
  - squaring the circle
  - Delos Problem (doubling the cube)
- but other tools are sometimes used, origamis for instance

Discussion (again)

- a construction plan can be seen from different points of view
- historically straightedge (ruler) and compass are considered as basic tools
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Discussion (again)

- a construction plan can be seen from different points of view
- historically straightedge (ruler) and compass are considered as basic tools
- but other tools are sometimes used, origamis for instance
  - With straightedge and compass one can solve polynomial equations of degree 2
  - With origamis one can solve polynomial equations of degree 3 and 4.

Discussion (again)

- a construction plan can be seen from different points of view
- historically straightedge (ruler) and compass are considered as basic tools
- but other tools are sometimes used, origamis for instance

RC-construction and education

Straightedge et compass construction are traditionally considered in math. exercises .

Definition

For points $\mathcal{B} = \{B_0, \ldots, B_m\}$, a point P is *RC-constructible* if there is a set of points $\{P_0, \ldots, P_n\}$ such that $P = P_n$ and every point P_i $(0 \leq i \leq n)$ is either a point from $\mathcal{B}$ or is obtained as the intersection of two lines, or of a line and a circle, or of two circles, themselves obtained as follows:

- any considered circle has its center in the set $\{P_0, \ldots, P_{i-1}\}$ and passing by some P_j;
- any considered line passes through two points from the set $\{P_0, \ldots, P_{i-1}\}$

A result about RC-constructibility

Decidability

There is an algorithm for deciding if a construction problem is RC-constructible if that problem can be expressed by polynomial equations on a computable field. (Galois, Lebesgue, Chen & Schreck, Gao & Chou, ...)

But ...

It is not easy to practically obtain a symbolic solution of such problems.
The Wernick's list is such an example (coming soon).

Algebraic results

Theorems

- a number α is RC-constructible if and only if there are $k+1$ fields
  - $\mathbb{F}_0, \ldots, \mathbb{F}_k$ ($\mathbb{F}_0$ is problem related)
  - $[\mathbb{F}_{i+1} : \mathbb{F}_i] = 2$
  - $\alpha \in \mathbb{F}_k$
- (Wantzel) If α is RC-constructible then its minimal polynomial has a 2^n degree. The converse is false.
- α is RC-constructible if and only if the splitting field of its minimal polynomial is an extension of degree 2^n over $\mathbb{F}_0$.

Method

The previous results suggest to use algebra to solve RC-construction problems:

A method based on algebra

- translation into algebra
- triangularization
- computation of the Galois group

The method can be applied in a two stages process

The method is applied first on a numerical witness to test RC-unconstructibility and ...
if this witness is RC-constructible the method is applied to the generic case.

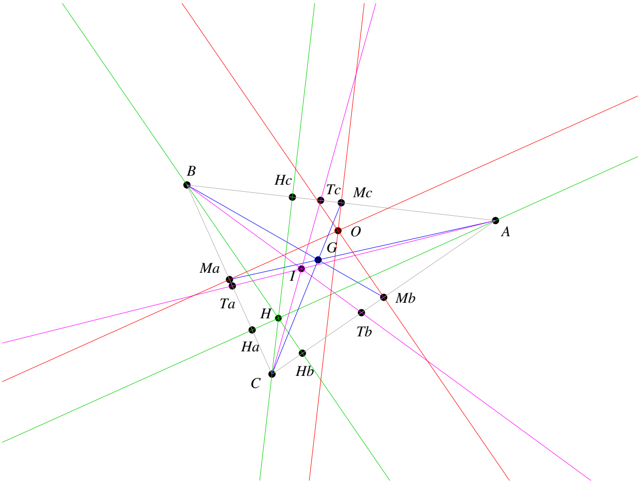
Wernick's list

Recently we systematically use this method in order to study a whole family of problems: the Wernick's list.

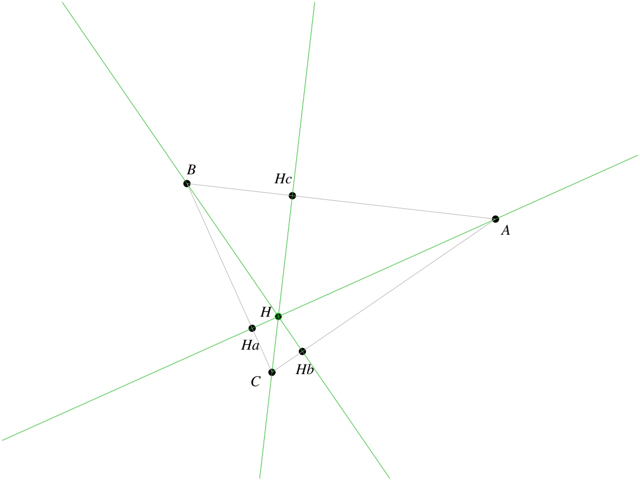
Wernick's problem (1982)

Given three located points selected from the set of 16 characteristic points of a triangle, is it possible to construct that triangle using only straightedge and compass? If yes, how to do it?

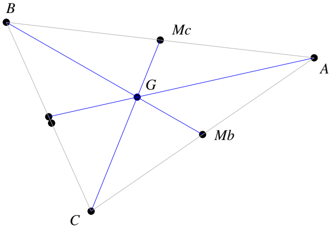
Characteristic points in a triangle



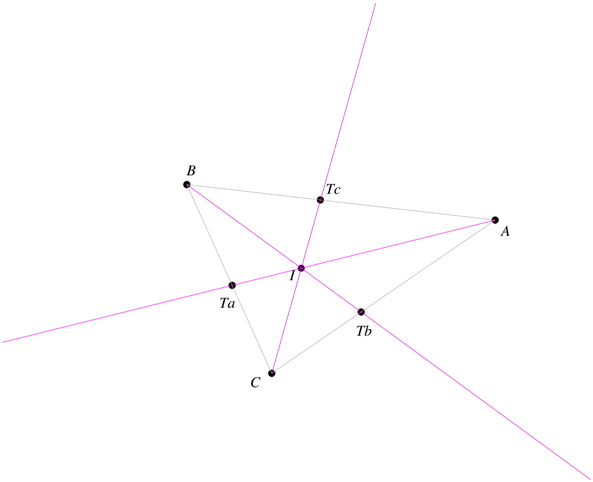
Characteristic points in a triangle



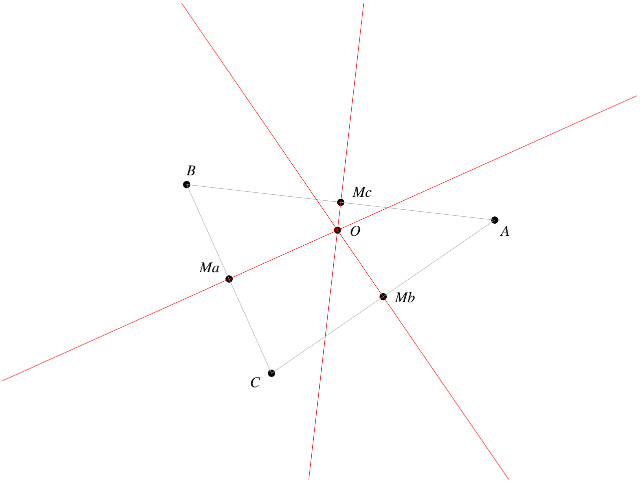
Characteristic points in a triangle



Characteristic points in a triangle



Characteristic points in a triangle



RC-construction and Wernick's list

Wernick's list:

- 139 different problems
- 15 were open problems (and one had a wrong status)
- 25 miss -constrained (R or L), 74 RC-solvable (S), 40 RC-unsolvable (U) (last open problems solved in 2014)

1. A, B, O	L	29. A, M_b, G	S	57. A, H, I		85. M_a, M_b, H_a	S	113. M_a, T_b, T_c	
2. A, B, M_a	S	30. A, M_b, H_a	L	58. A, T_a, T_b		86. M_a, M_b, H_c	S	114. M_a, T_b, I	
3. A, B, M_c	R	31. A, M_b, H_b	L	59. A, T_a, I	L	87. M_a, M_b, H		115. G, H_a, H_b	
4. A, B, G	S	32. A, M_b, H_c	L	60. A, T_b, T_c	S	88. M_a, M_b, T_a		116. G, H_a, H	S
5. A, B, H_a	L	33. A, M_b, H	S	61. A, T_b, I	S	89. M_a, M_b, T_c		117. G, H_a, T_a	S
6. A, B, H_c	L	34. A, M_b, T_a	S	62. O, M_a, M_b	S	90. M_a, M_b, I		118. G, H_a, T_b	
7. A, B, H	S	35. A, M_b, T_b	L	63. O, M_a, G	S	91. M_a, G, H_a	L	119. G, H_a, I	
8. A, B, T_a	S	36. A, M_b, T_c	S	64. O, M_a, H_a	L	92. M_a, G, H_b	S	120. G, H, T_a	
9. A, B, T_c	L	37. A, M_b, I	S	65. O, M_a, H_b	S	93. M_a, G, H	S	121. G, H, I	
10. A, B, I	S	38. A, G, H_a	L	66. O, M_a, H	S	94. M_a, G, T_a	S	122. G, T_a, T_b	
11. A, O, M_a	S	39. A, G, H_b	S	67. O, M_a, T_a	L	95. M_a, G, T_b		123. G, T_a, I	
12. A, O, M_b	L	40. A, G, H	S	68. O, M_a, T_b		96. M_a, G, I		124. H_a, H_b, H_c	S
13. A, O, G	S	41. A, G, T_a	S	69. O, M_a, I	S	97. M_a, H_a, H_b	S	125. H_a, H_b, H	S
14. A, O, H_a	S	42. A, G, T_b		70. O, G, H_a	S	98. M_a, H_a, H	L	126. H_a, H_b, T_a	S
15. A, O, H_b	S	43. A, G, I		71. O, G, H	R	99. M_a, H_a, T_a	L	127. H_a, H_b, T_c	

Unconstructibility

example Wernick # 77

Statement. The task is to construct triangle (A, B, C) with straightedge and compass from three points O, H_a and T_b.

```
F := [Oo(0, 0), Ha(-1, -3), Tb(-3, 0)];
```

$$\begin{cases} x_A^2 - x_B^2 + y_A^2 - y_B^2 = 0 \\ x_A^2 - x_B^2 + y_A^2 - y_B^2 = 0 \\ (-1 - x_B)(y_C - y_B) + (-3 - y_B)(x_B - x_C) = 0 \\ (x_A + 1)(x_C + 1) + (y_A + 3)(y_C + 3) = 0 \\ (-3 - x_A)(y_C - y_A) - y_A(x_A - x_C) = 0 \\ \mathrm{onb}(-3, 0, x_B, y_B, x_A, y_A, x_C, y_C) = 0 \end{cases}$$

After fixing the order $x_C, y_C, x_B, y_B, x_A, y_A$, we get by triangularization (and filtering):

$$84349y_A^8 + 668100y_A^7 + 908434y_A^6 - 6940782y_A^5 - 32743501y_A^4 - 63643476y_A^3 - 72253168y_A^2 - 56499066y_A - 25568010$$

Unconstructibility

example Wernick # 77

Statement. The task is to construct triangle (A, B, C) with straightedge and compass from three points O, H_a and T_b.

$$84349y_A^8+668100y_A^7+908434y_A^6-6940782y_A^5-32743501y_A^4-63643476y_A^3-72253168y_A^2-56499066y_A-25568010$$

```
galois(st[1][6]);
     "8T50", {"S(8)"}, "-", 40320,
     {"(1 8)", "(2 8)", "(3 8)", "(4 8)",
      "(5 8)", "(6 8)", "(7 8)"}
```

Unconstructibility

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      "(5 8)", "(6 8)", "(7 8)"}
```

Result. This problem is not RC-constructible (a counter example is enough).

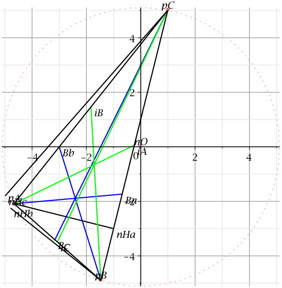
Unconstructibility

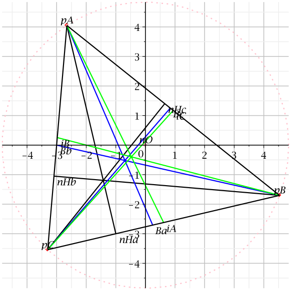
example Wernick # 77

Statement. The task is to construct triangle (A, B, C) with straightedge and compass from three points O, H_a and T_b.

$$84349y_A^8+668100y_A^7+908434y_A^6-6940782y_A^5-32743501y_A^4-63643476y_A^3-72253168y_A^2-56499066y_A-25568010$$

Result. This problem is not RC-constructible ... but it has two real solutions.





Constructibility

Example Wernick # 108

Statement. The task is to construct triangle ABC knowing points T_a (foot of the inner-bisector form A), H_h and M_a.

Contrarily to the result found in the literature, we found that this problem is RC-constructible:

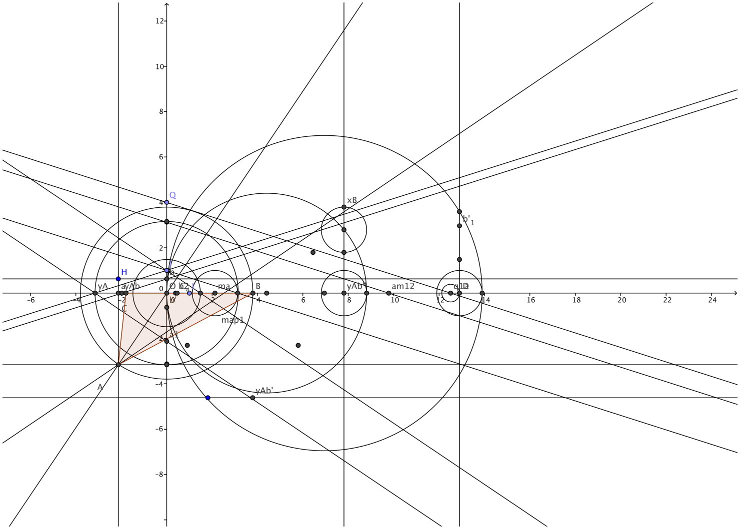
```
F := [Ta(0, 0), Hh(1, 0), Ma(a, b)]
```

After triangularization and filtering:

$$\begin{cases} x_C - 2a + x_B = 0 \\ -ay_B + bx_B = 0 \\ y_C - 2b + y_B = 0 \\ (a^2 + b^2)y_B^2 + (-2a^2 * b + ab * x_A - 2b^3 \\ \qquad + b^2 y_A - ab)y_B + 2ab^2 - x_A b^2 = 0 \\ ax_A + by_A - a = 0 \\ (a^4 + 2a^2 b^2 + b^4)y_A^2 + (-2a^3 b - 2ab^3 - a^2 b)y_A - a^4 \\ \qquad + a^2 b^2 + a^3 = 0 \end{cases}$$

Constructibility

RC-construction of # 108 with Geogebra



Constructibility

Example Wernick # 119

We also solved the RC-constructibility status of problem #119 (unsolved before).

Statement. The task is to construct a triangle ABC knowing I (incenter), H_a and G. We test the example

```
S = [Ii(0,0),Ha(1,-2),G(1,1)].
```

And we get:

$$P(y_C) = 289y_C^4 - 867y_C^3 - 57528y_C^2 - 99144y_C - 41472$$

which is RC-solvable.
So, we try

```
S = [Ii(0,0),Ha(1,-2),G(a,b)]
```

and we get a 400 lines polynomial (!) whose Galois'group has order 4. *This problem is RC-constructible.*

Constructibility

Construction of Wernick # 119

I used the Gao & Chou's method to solve this problem ... and have a construction in the previous style. But the expressions are too huge to be processed this way:

$$12754584a^{13} + 76527504a^{11}b^2 + 191318760a^9b^4 + 255091680a^7b^6 + 191318760a^5b^8 + 76527504a^3b^{10} + 12754584ab^{12} - 55269864a^{12} - 280600848a^{10}b^2 - 573956280a^8b^4 - 595213920a^6b^6 - 318864600a^4b^8 - 76527504a^2b^{10} - 4251528b^{12} + 93533616a^{11} + 416649744a^9b^2 + 731262816a^7b^4 + 629226144a^5b^6 + 263594736a^3b^8 + 42515280ab^{10} - 72748368a^{10} - 314613072a^8b^2 - 515852064a^6b^4 - 387361440a^4b^6 - 121877136a^2b^8 - 8503056b^{10} + 18108360a^9 + 115263648a^7b^2 + 214465968a^5b^4 + 155574432a^3b^6 + 38263752ab^8 + 8030664a^8 - 4408992a^6b^2 - 48813840a^4b^4 - 42200352a^2b^6 - 5826168b^8 - 4269024a^7 - 11547360a^5b^2 + 1469664a^3b^4 + 9867744ab^6 - 536544a^6 + 2309472a^4b^2 + 2869344a^2b^4 - 1469664b^6 + 355752a^5 + 618192a^3b^2 - 390744ab^4 + 55080a^4 - 89424a^2b^2 - 71928b^4 - 9936a^3 - 24624ab^2 - 3024a^2 - 1296b^2 - 264a - 8$$

Temporary conclusion

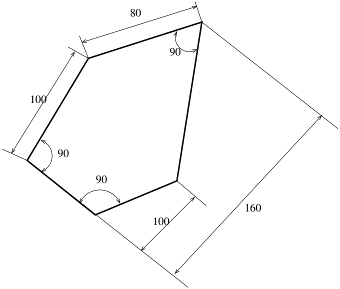
Geometric methods provide readable constructions but they are not complete and unable to detect RC-unconstructibility.

Algebraic methods are good in deciding if a problem is RC-solvable or not, but they do not provide effective constructions

Back to CAD

Well-constrainedness and invariance

Consider our first example:



Well-constrainedness and invariance

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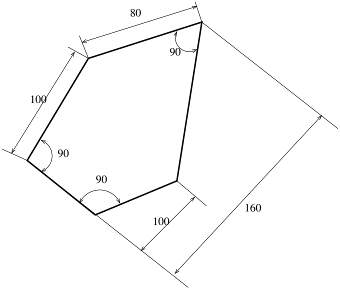

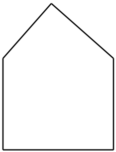

a particular solution, but ...

Well-constrainedness and invariance

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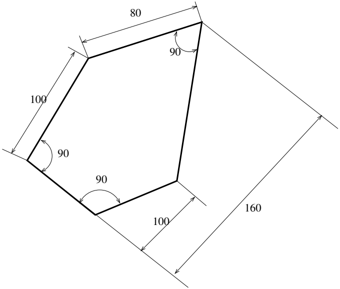

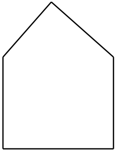

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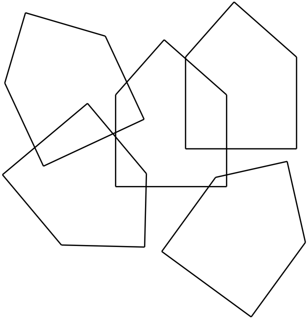
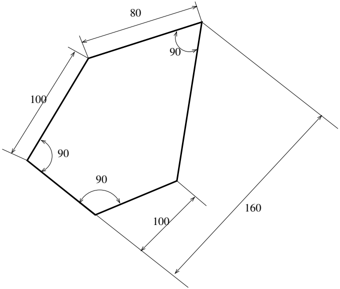


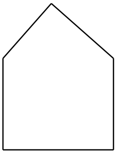
there are infinitely many solutions

Well-constrainedness and invariance

Consider our first example:



This system is under-constrained



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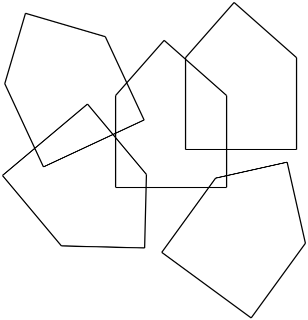

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Well-constrainedness and invariance

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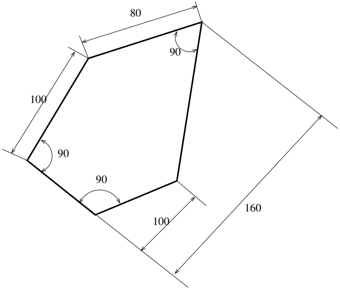

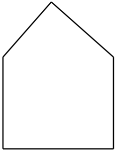

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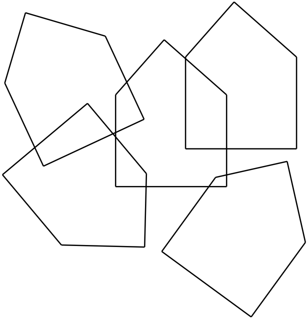


there are infinitely many solutions

This system is under-constrained

All the solutions come from a finite number of them by using isometries

Some definitions

Invariance

A constraint system is invariant under the action of a group G iff its set of solutions F is invariant by any $g \in G$, that is $g(F) = F$. F is the union of the orbits under the action of G

Rigidity

A constraint system invariant by isometries is said rigid or iso-rigid or well-constrained modulo the isometries iff its solution set is the union of a finite number of orbits.

Reference

A reference is a constraint system such that isometries simply transitively act on its solution set.

Rigidity theory

Cauchy's theorem

Two convex polyhedrons whose faces are isometric in pairs are isometrics.

Rigidity theory

Cauchy's theorem

Two convex polyhedrons whose faces are isometric in pairs are isometrics.

Framework

A framework is a triple $F = (V, E, p)$ where $G = (V, E)$ is a graph and p a mapping of the vertices to $\mathbb{R}^d$.

Rigidity theory

Cauchy's theorem

Two convex polyhedrons whose faces are isometric in pairs are isometrics.

Framework

A framework is a triple $F = (V, E, p)$ where $G = (V, E)$ is a graph and p a mapping of the vertices to $\mathbb{R}^d$.

Rigidity?

- unique "realization" modulo isometries = strong rigidity,
- local uniqueness = rigidity
- parameters and neighborhood = generic rigidity
- infinitesimal motions = infinitesimal rigidity

Rigidity theory

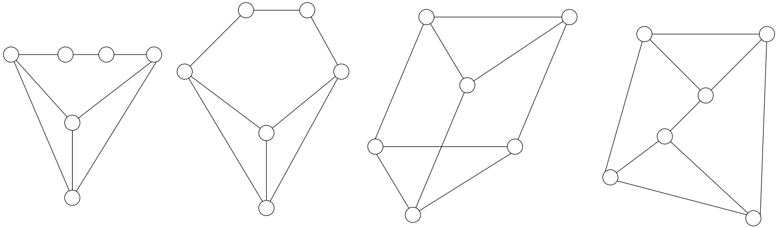
Cauchy's theorem

Two convex polyhedrons whose faces are isometric in pairs are isometrics.

Framework

A framework is a triple $F = (V, E, p)$ where $G = (V, E)$ is a graph and p a mapping of the vertices to $\mathbb{R}^d$.

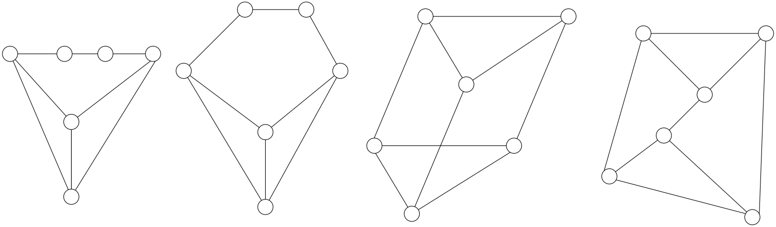
Rigidity?



Infinitesimal motions

Definition

Let $F = (V, E, p)$ be a framework, an infinitesimal motion of F is a mapping m of V into $\mathbb{R}^d$ such that
$(m(a) - m(b))(p(a) - p(b)) = 0$ for every edge (a, b) of E
An infinitesimal flexion is an infinitesimal motion which is not a direct isometry.
A framework is infinitesimally rigid if it does not admit an infinitesimal flexion.



Results (1)

Implications

- strong rigidity $\Rightarrow$ rigidity
- infinitesimal rigidity $\Rightarrow$ rigidity
- infinitesimal rigidity $\Rightarrow$generic rigidity

Gluck's theorem

If a graph G has a generically rigid realization then almost all its realizations are rigid.

Laman's theorem

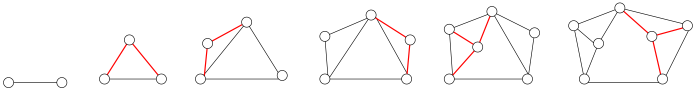
In 2D, a graph with n vertices is generically rigid iff it has $2n - 3$ edges and for all sub-graph containing k edges related to m vertices, we have $k \leqslant 2m - 3$.

Results (2)

Henneberg's constructions

Henneberg's rules

- add a new veretx connected by two new edges
- divide an edges into two by adding a vertex wich is also connected to a third vertex



Theorem

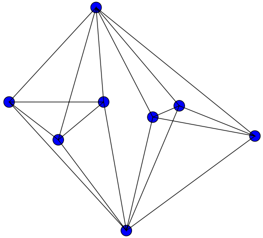
Every Laman graph can be constructed from K_2 by using Henneberg's rules.

Generalization?

- The 3D version of Laman assertion is not a theorem.
- There is no known combinatorial characterization of generic rigidity of graphs
- There is no known generalization of Henneberg's rules in 3D.

Generalization?

- The 3D version of Laman assertion is not a theorem.



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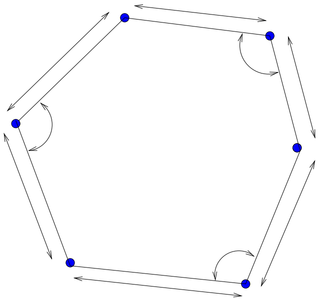
Invariance and decomposition

When a constraint system is invariant by isometries, the decomposition method are based on rigidity criteria. Indeed, the Dulmange-Mendelsohn cannot be directly used:

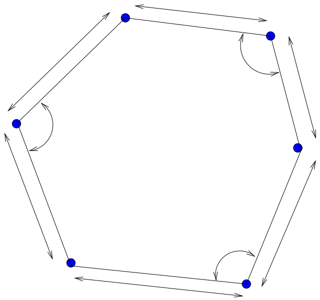
- either you fix a reference (and you destroy the symmetry of the problem)
- or you find a method compatible with that invariance

Example



Bottm-up: minimal rigid sub-systems

Sunde, Hofmann et al.: aggregation rules of CD-sets (clusters)



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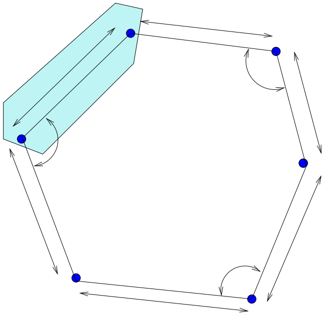

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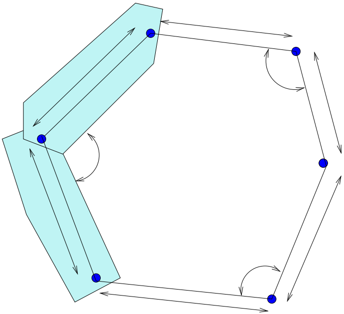

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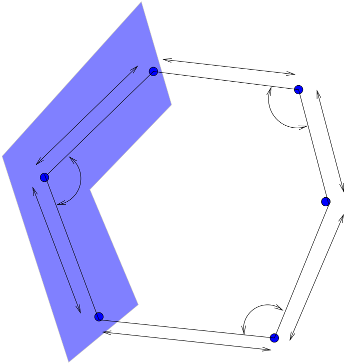

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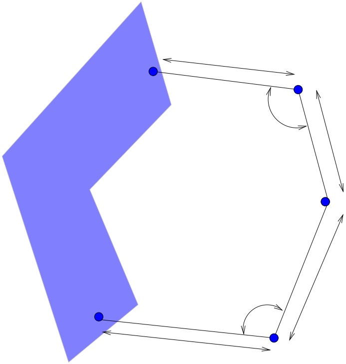

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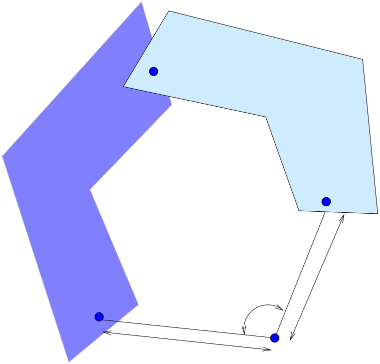

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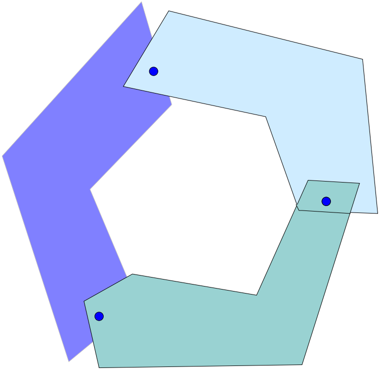

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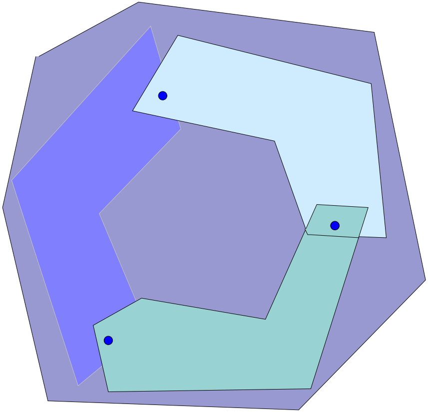
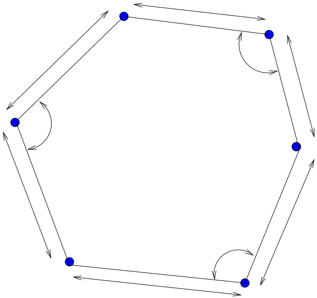


Top-down: "maximal" rigid sub-system

Owen: disaggregation rules (articulation pairs)



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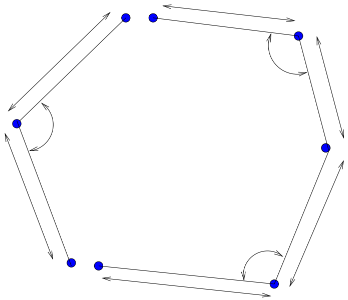

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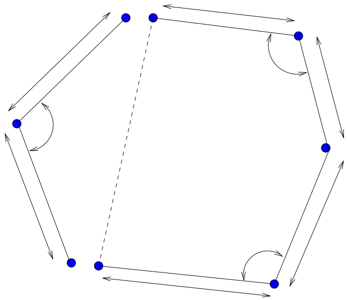

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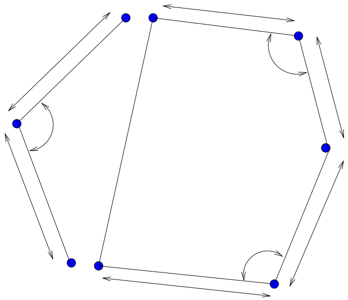

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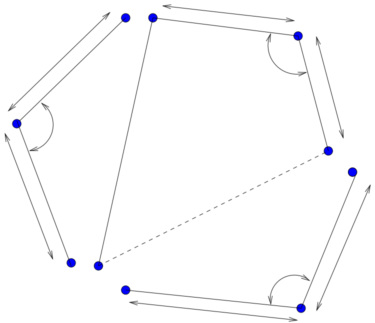

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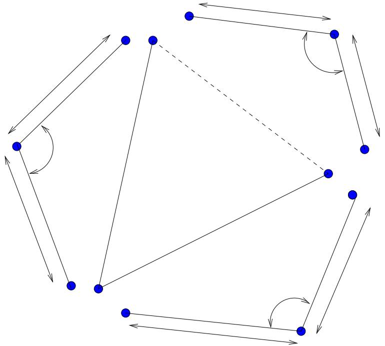

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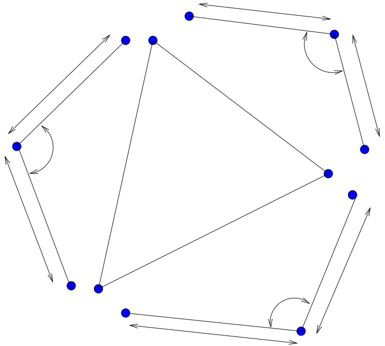

Partial conclusion

▷ Generalization to other groups: similarities, homographies ...

▷ high level pattern recognition: distinct objects, articulations ...

Constructions and Proofs in Geometry

A construction as a witness

Intuitionist proof.

Solving a construction problem consists in proving a $\forall\exists$ theorem in an intuitionist way

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Constructibility

For a given constraint system $\mathcal{C}(\mathcal{X}, \mathcal{A})$, with unknowns $\mathcal{X}$and parameters $\mathcal{A}$, prove

$$\forall \mathcal{A} \exists \mathcal{X}, \mathcal{C}(\mathcal{X}, \mathcal{A})$$

Construction

For a given constraint system $\mathcal{C}(\mathcal{X}, \mathcal{A})$, with unknowns $\mathcal{X}$and parameters $\mathcal{A}$, find F such that,

$$\forall \mathcal{A}, \forall \mathcal{X}, \mathcal{C}(\mathcal{X}, \mathcal{A}) \Leftrightarrow \mathcal{X} = F(\mathcal{A})$$

A construction as a witness

Intuitionist proof.

Solving a construction problem consists in proving a $\forall\exists$ theorem in an intuitionist way

$$
\begin{array}{l}
\forall \mathcal{A} \forall \mathcal{X}. \\
\left[\begin{array}{l}
\mathcal{C}(\mathcal{X}, \mathcal{A}) \\
\Leftrightarrow \\
\left(\begin{array}{l}
\quad (\delta_1(\mathcal{A}) \supset \mathcal{X} = F_{1,1}(\mathcal{A}) \vee \ldots \vee \mathcal{X} = F_{1,k_1}(\mathcal{A})) \\
\wedge (\delta_2(\mathcal{A}) \supset \mathcal{X} = F_{2,1}(\mathcal{A}) \vee \ldots \vee \mathcal{X} = F_{2,k_2}(\mathcal{A})) \\
\ldots \\
\wedge (\delta_l(\mathcal{A}) \supset \mathcal{X} = F_{l,1}(\mathcal{A}) \vee \ldots \vee \mathcal{X} = F_{l,k_l}(\mathcal{A})) \\
\wedge \ (\Delta(\mathcal{A}) \supset \ \Psi(\mathcal{X}, \mathcal{A})) \\
\wedge \ (\Omega(\mathcal{A}) \supset \bot)
\end{array} \right)
\end{array} \right] \\
\wedge(\delta_1(\mathcal{A}) \vee \ldots \vee \delta_l(\mathcal{A}) \vee \Delta(\mathcal{A}) \vee \Omega(\mathcal{A}))
\end{array}
$$

Tacking degenerate cases into account

A simple example

Suppose that you can deduce that point A is on line L_1 and on line L_2, is it possible to infer that A can be constructed by $interll(L_1, L_2)$?

- if you can prove that L_1 and L_2 are not parallel, then it is OK;
- if you can prove that L_1 and L_2 are parallel, it is also OK: you cannot construct A that way but you got a precious information $L_1 // L_2$;
- if you cannot prove any of the two statements, you have to consider two cases and maybe two different constructions.

Discover a theorem and rigidity

a qualitative study of a set of relations seems useful in theorem discovery. It allows to get

- the connected part
- the articulated part (?) using rigidity
- the over-constrained parts of a set of relations

Constructive statement

Most of the method in automated reasoning in geometry are based on constructive statement.

- triangularization methods are sensible to the order of variables.
- if a construction (not forcefully a RC-construction) is known, the triangularization is easier to perform.
- RC-constructions are difficult to obtain in general.

Conclusion

- proofs and constructions in geometry are closely related
  - a constructibility theorem is a special case of theorem
  - automatic methods are often based on constructive statements
- it is usual to consider proofs within a construction process
- I think it could be useful to consider constraint systems decomposition techniques in theorem discovery.

Thanks

感謝您的關注
你有問題嗎？